THE PHILOSOPHY OF SYMBOLIC FORMS

VOLUME THREE: THE PHENOMENOLOGY OF KNOWLEDGE

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Chapter 5

The Foundations of Scientific Knowledge

1. Empirical and Constructive Manifolds

THE REALM OF NUMBERS shows us, in typical purity and perfection, the example of a sphere of objects which takes form from an underlying original relation, and which may be wholly surveyed and determined through this relation. Thought starts from a pure relation, which at first seems to be of the simplest conceivable form—which comprises nothing other than an ordering of intellectual elements through a law of sequence that is imposed on them. But from this elementary law flow ever broader and more complex determinations, which in turn are interwoven in a strictly lawful way, until the totality of these connections gives rise to the aggregate of the "real numbers," in which the wonderful edifice of analysis is grounded. Here there is never any danger of reaching absolute bounds or internal contradictions as long as mathematical knowledge remains true to its own constructive principle—as long as it admits of no other objects than those which it can gain and derive directly from this principle. It is the fundamental form of the relation itself which posits and marks off a determinate sphere of objects, and which in this determination makes them into a totality which knowledge can master in theory.

But this kind of intellectual mastery seems to stop as soon as we pass beyond the sphere of the mathematical—as soon as we take the step from the ideal to the real. For here begins the realm of matter as opposed to that of pure form. In place of an original unity, which lawfully unfolds and explicates itself into a multiplicity, we now have a manifold which lies before us purely as such, an existing multiplicity. This manifold—at least in its immediate presentation—is not "constructible": we must accept it as simply given. Precisely this givenness seems to be the specific characteristic which distinguishes the physical from the purely mathematical. Here it is not an objective world that is built up for us with inner consequence and consistency but, rather, an outward existence that is given to us through the intermediary of sensations and sensory intuition. And this form of "giving" can only be fragmentary. We must move—not according to a predetermined plan but according to the dictates of planless, accidental observation—from one point in this existence to another. And we may be well pleased if, at the end of our road, we can connect all these points into a line whose form can be described and expressed in general terms. At every moment we must be prepared to replace this form by another whenever there is a change in the data on which our intellectual synthesis is based.

In the face of such empirical constraint, theoretical thinking seems powerless. *Natura non vincitur nisi parendo;* it is not by forcing its universal form on nature that thought can gain access to the physical world, but only by immersing itself in its particular structures and seeking to copy them trait for trait. Our horizon is no longer sharply delimited, marked off from the start; on the contrary, every extension of the contents seems to compel a change in the modality of our view, a shift in our line of sight. What we call "nature," what we call the "existence of things," confronts us at first as a mere rhapsody of perceptions. It may be possible to line up these perceptions as on a string and describe them in their coexistence and succession; but this mode of listing remains radically different from the fundamental form of series that we found expressed in the progression of the whole numbers. For here when one member follows another, this does not mean that it follows from that other, that the second can be derived from the first in accordance with a universal rule that can be determined for the whole of the series. The development, the progression, the method has become a mere empirical succession. But this implies a fundamental change in the relation between individuality and universality. Each single number is also an individual concept, an object with determinations and characteristics pertaining to it alone. This individuality does not belong to it as such, but only in the system of numbers. Here, then, the individual is conceived as a pure positional value.

But the individual perception aspires to be something more than a mere position in a series. It stands, one might say, for itself and on itself, and its meaning resides precisely in this particularity. Of course it, too, is articulated with the totality which we designate as space and time. But it fills this totality, it fills the particular point of space and moment of time in which it is situated with a unique content, which cannot be reduced to a mere determination of "where" and "when." And herein lies the character of the mere "datum." Each perception is given only to one observer, under his special spatio-temporal conditions. It is by no means self-evident—indeed it is incomprehensible at first sight—how the perception can emerge from this isolation, how it can be connected with other perceptions. For such a connection seems to demand a grouping together of elements which must be regarded not only as accidental but as fundamentally dissimilar. Without its inherent heterogeneity, perception would not seem to be perception, for then it would lose the qualitative particularity pertaining to its very essence; and with this heterogeneity it seems to defy the form of the system, which is an indispensable condition for all knowledge and theoretical understanding.

This antinomy contains the dialectical germ of all scientific concept formation. For when thought passes from the sphere of mathematical objects to that of physical objects, it does not, of course, cast off its own form and its own presuppositions; on the contrary, it seeks confirmation of these presuppositions in the resistance presented by the datum. And through this resistance thought discovers in itself a new power which had hitherto seemed locked within it. It now aspires to perform the impossible: to treat and regard the given as though it were not alien to thought, as though it were posited by thought itself and produced by its constructive conditions. The form of the merely factual manifold, in which perception first presents itself, must now be transposed into the form of a conceptual manifold. Concrete physical thinking, as it operates in the history of natural science, does not inquire whether such a transposition is possible but directly turns the problem into a postulate. It translates the conceptual aporia into an act. All scientific concept formation begins with such an intellectual act. Thought preserves its discursive nature not by contenting itself with the order of the given but by striving actually to "run through" this series. And this it can do only by seeking a rule of transition that will lead from one link to another. This rule, which is not immediately given but is solely postulated and sought, remains the characteristic by which the peculiar "facticity" of scientific thinking differs from every other form of mere factual knowledge. Even the *vérités de fait* that are disclosed and established in physical thinking are determined by the special character of the physical ratio and are impregnated with it.

This becomes strikingly evident as soon as we compare the facts of physics with those of another field, such as history for example. Here we find immediate confirmation of Goethe's saying: "the highest thing would be . . . to recognize that everything factual is itself theoretical." There is no such thing as a sheer facticity, as an eternal and immutable datum: on the contrary, what we call a fact must always be theoretically oriented in some way, must be seen in reference to a definite conceptual system, which implicitly determines it. The theoretical means of determination are not subsequently added to the sheer fact but enter into the definition of the fact itself. Thus the facts of physics are distinguished at the very outset from those of history by their specific intellectual perspective. "Carlyle says somewhere," remarks Henri Poincaré in *La Science et l'hypothèse,* "that the fact alone is decisive. John Lackland has passed by here; that is noteworthy, that is a reality for which I would give all the theories in the world. That is the language of the historian. The physicist would say on the contrary: John Lackland has passed by here; that is a matter of indifference to me, because he will not pass by again." [1] This pregnant formulation shows us the fundamental contrast between the two basic meanings of facticity. Even where the physicist describes a single event, confined to a definite situation in space and moment in time, he is not concerned with the particular as such, but considers it under the aspect of its repeatability. What he wants to establish is not that something has happened here and now; rather, he is interested in the conditions of the occurrence. His question is whether, under these same conditions, the same occurrence will be observed at other places and times, or how it will change under determinate variations of these conditions. Even where a single fact is investigated and confirmed, the inquiry aims not at this fact alone but at the rule according to which it is conceived as recurring. At the start the form of this rule is still an open question, and we must take care not to make any definite statement about it prematurely.

There was a period in physics when it looked as though this form had been definitively established. In his epoch-making article "On the Conservation of Energy" (1847) Helmholtz set forth the universal theory of causality as this original form of physical thinking. For him it is the *conditio sine qua non* for the formulation of the scientific problem, the condition for the "comprehensibility of nature." In the present state of physical knowledge, epistemology must judge more cautiously. The ques-

1. Poincaré, *La Science et l'hypothèse,* p. 168.

tion of whether all explanation of nature must necessarily lead to causal laws of a definite type, or whether it must content itself with mere laws of probability cannot, however it may be decided, be resolved by a simple edict of thought. Here only immersion in the conceptual order of physics itself can bring a decision, can teach us how, within scientific thinking, the realm of the purely dynamic laws can be marked off from that of mere statistical laws.[2] But even where physical thinking does not lay claim to understanding a process in a strictly causal sense but contents itself with establishing statistical rules, its essential goal is not the occurrence itself but its regularity. And the judgment which establishes this regularity can never be dissolved into a mere sum, an aggregate of statements about particular instances.

Of course, in accordance with its basic trend, strict empiricism must attempt such a dissolution. Mach, for example, seems to regard the law of gravity as nothing more than the summation of a large number of concrete observations which undergo no change in the summation except that they are given a common linguistic expression. Here Galileo's law is regarded solely as an abbreviated expression of a table in which certain individual values of *t* are correlated with certain individual values of *d*. The only reason why we choose a general formula instead of applying this table explicitly to all hitherto observed cases must be sought in the economy of thought, which demands that we make the most sparing use of signs. But our formula takes on meaning only if we replace the indeterminate variables with definite numerical values.

If this view held good, the world of physical facts would be reduced to a mere historical facticity: the difference between the two would reside not in the facts themselves but solely in the signs we employ to represent them. But even if we follow radical empiricism in that view, a new question arises at this point within the frame of our universal problem. The philosophy of symbolic forms has shown us everywhere that the "sign" is never a merely accidental and outward garment for the thought, but that the use of the sign represents a basic tendency and form of thought itself. We must ask, then: what tendency in physical thought is it that compels us to single out a determinate sign language, the sign language of the mathematical formula, and favor it over all others? In view of all the

2. Cf. esp. M. K. E. L. Planck, *Dynamische und statistische Gesetzmässigkeit* (Berlin, 1914); reprinted in *Physikalische Rundblicke, Gesammelte Reden und Aufsätze* (Leipzig 1923), pp. 82 ff.

insight we have gained in regard to language and its spiritual constitution, we can no longer suppose that mere grounds of convenience prevail here; we cannot but suppose a deeper and more intimate relation between the form of thought and the form of language. Whether this presumption or, if you prefer, this systematic "prejudice," is confirmed can be shown, of course, only by a penetrating analysis of concept formation and signification in physics. Here again the way leads from an understanding of the signs to an understanding of the things, of what is designated; an investigation and analysis of the symbols in which physical judgments are expressed and in which they first take on their appropriate form, will, it is to be hoped, make intelligible the modality and character of physical objectivity.

It was the merit of Pierre Duhem that he first took this road, in his book on physical theory. With extraordinary sharpness this work discloses all the ideal mediations through which we must pass if we are to gain physical theorems and judgments from a mere observation of individual phenomena. Duhem shows that it is the construction of a determinate world of symbols that first opens up access to the world of physical reality. And each of the symbols here created presupposes the original symbol of "real number" as its actual foundation.[3] What appears at first sight as a purely factual manifold and as a factual diversity of sense impressions takes on physical meaning and value only when we reproduce it in the realm of number. Of course we shall not do justice to this reproduction and the highly complex formal law under which it stands if we take it in a purely material sense, if we start from the assumption that to enter into the world of physics, it suffices to substitute contents of a different mode and imprint for the contents given in perception. With each perceptual class we should then merely have to correlate a particular substrate, which would be the complete expression of its genuine, its truly physical reality. What presents itself to the senses as a feeling of warmth would then be recognized in its physical truth as molecular motion; what is given to the eye as color would be defined as vibration of the ether. But this mode of transference, in which the content of immediate perception is transformed, piece by piece as it were, into another, mediated content, is far from exhausting the meaning of physical method. Physical method is concerned, far more, with relating sensory phenomena—colors, tones, tactile sensa-

3. In regard to Duhem's theory of physical objectivity see above, pp. 21–2; also *Substanzbegriff und Funktionsbegriff*, 2d ed., pp. 189 ff.

tions, etc.—as a whole, to another, intellectual standard and thereby elevating them to a new dimension. In principle we can never compare the particular sensation with its determinate objective-physical substrate; what can be compared is, rather, on the one hand the totality of the phenomena of observation, and on the other hand the total system of concepts and judgments in which physics expresses the order and lawfulness of nature; and here we can measure the one by the other. There was a period in the history of physics, when it was believed that we could overcome scientific materialism by replacing the notion of a unitary, fundamental substance with another conception that was also taken substantially. Substantial matter was replaced by substantial energy or substantial ether. But no really new deepening of the critique of knowledge was ever achieved in this way.

A new perspective opened only when the concept of the physical copy was analyzed more closely as to its meaning and achievement. For now it became evident that the copy never leapt immediately from an element in the perceptive series to an element in the series of physical concepts, so that the two could be examined for direct similarity or correspondence. Rather, such a correspondence may be sought only between the totality of the data of empirical observation and the totality of the theoretical concepts, physical laws and hypotheses. It is through an increasing awareness of this relationship and its logical consequences that modern physics has overcome materialism, not only in an ontological sense but in a more comprehensive methodological sense. More and more it abandoned the explanation of natural phenomena, which consisted solely in replacing certain groups of concrete sensuous phenomena by their abstract geometrical representatives or mechanical models. The turn away from this form of explanation seemed to mean only a step toward the positivism which regards physical laws as nothing more than a description of the natural process, but this was not so. The difference became evident as soon as physicists, instead of stressing only the negative factors, turned toward positive determination and reflected on the specific character of their means of description. These means are far removed from the mode of facticity which positivism regards as the sole criterion of reality: they belong to the same sphere as the structures of pure mathematical thinking.

The recognition of this original duality is the necessary condition for an understanding of the harmony which the scientific concept demands and institutes. This harmony signifies something fundamentally different from mere agreement: it is a genuinely synthetic act, which links opposites together. Such a synthesis of opposites is included in every genuine physical concept and in every genuine physical judgment. For physical concepts and judgments are always concerned with relating two forms of manifold and, one might say, permeating them with each other. The starting point is a merely empirical, given manifold: but the aim of theoretical concept formation is to transform such a manifold into a rationally surveyable, "constructive" multiplicity. This transformation is never concluded—it is always begun anew, with increasingly complex means. The basic epistemological question concerning the possibility of applying mathematical concepts to nature goes back ultimately to this relationship and the problem it comprises. The difficulty is that such a transference seems possible only on the basis of a conscious *μετάβασις εἰς ἄλλο γένος*, that it may be said to force the phenomena into an order other than that to which they originally belong.

Yet if we take the standpoint not of a realistic metaphysics but of the philosophy of symbolic forms, this transformation loses a large part of its paradoxical character. For we have shown from the most diverse angles that all intellectual life and development operate through transformations and metamorphoses of this kind. The very beginning and possibility of language were conditioned by such a metamorphosis, for language cannot simply designate given impressions or representations: the sheer act of naming always comprises a change of form, an intellectual transposition. We have seen that this transposition becomes more and more pronounced as language progresses, as it comes into its own. Gradually language breaks away from its confinement to the given, its similarity to the given: from the phase of mimetic and analogical expression it progresses to purely symbolic formation. Scientific knowledge repeats the same process in a different dimension. It, too, gains approximation to nature only by learning to renounce it, by moving the given into an ideal distance. Accordingly, the problem here does not lie in this removal, in this intellectual positing of distance; our problem is, rather, to determine the special direction in which the work of physical thinking progresses, and to distinguish it sharply from other fundamental trends of formation. And we shall gain an insight into this difference not only by considering the universal goal toward which physical thinking strives but, in addition, by breaking down the road that leads to it, into its separate stages. We shall have to retrace this road step by step, for only in this way shall we be able to describe it. In speaking of the portrayal of great men Goethe once said that the source can only be described as it flows. The same is true of any living movement of the human spirit. The nature of its progress cannot be described solely in abstract formulas, but must be grasped in its actuality, in the energy of the movement itself. The methodological law of development cannot be explained except through the concrete process, its beginning and subsequent development, its intellectual crises and vicissitudes.

Dogmatic empiricism and dogmatic rationalism both end in failure, because they cannot do justice to this actuality, this pure process-character of knowledge. They negate the process by denying polarity, which is the true driving force of knowledge, the very principle of its movement. This polarity is destroyed if, instead of relating the opposing factors to one another and connecting them intellectually, we seek to reduce the one to the other. Empiricism does this by dissolving the constructive concepts in the given; rationalism, conversely, does it by reducing every datum to the form of its conceptual determination. But in both cases we have a leveling of the fundamental oppositions whose clash truly builds up the objective world of physical knowledge. Fruitful correlativity is replaced by naked coincidence. This is to neglect the creative factor in the concept and in experience as well, for concept and experience develop the forces inherent in them only by measuring themselves one against the other. The succession of perceptions, the empirical series of coexistence and succession, present the question that is to be solved by means of the conceptual, constructive series. Experience sets up a coexistence and succession that are to be transposed progressively into a unity. A totality of members *a, b, c, d* . . . which are at first given solely in their "thatness," in the actuality of their spatio-temporal togetherness, are to be recognized as belonging together, are to be linked by a rule on the basis of which the production of the one from the other can be determined and foreseen. This law of production is never immediately given in the same way as perceptions: it must be injected into them in a purely intellectual, hypothetical way. We attempt to order the elements *a, b, c, d* . . . in such a way that they can be thought of as members in a series x_1 x_2 x_3 x_4 . . . which is characterized by a determinate "universal member." When particular magnitudes are appointed for this universal member, the individual case "results" in the strictest sense.

But this result never exists absolutely: it must always be gained and ascertained anew through increasingly refined methods of series formation. The relating of the empirical series form to the mathematical, ideal form is a process that never ceases. But on the other hand, the one never passes wholly into the other: each retains its distinct structure. In this connection we again perceive how the mathematical-physical concept begins with experience but does not arise from it. Experience comes first, in the sense that it formulates the problem. But we cannot expect it to solve the problem; the solution must follow from the basic trend of mathematical-constructive thinking. Platonically speaking, perception is and remains the paraclete of physical thought, but it does not produce the powers that it awakens. The objective world of physics arises and is affirmed through such an interplay of forces. Time and time again the contact with empirical intuition and its immediate reality has led the mathematical-physical concept to its own development, compelling it to yield up its own latent possibilities. Yet in this process of self-unfolding the concept is soon carried beyond the limits of the initial question. It not only provides a framework for present empirical problems; it also reaches out into the future; it prepares the intellectual instruments for possible experience and points the way by which this purely theoretical possibility can be translated into actuality.

This twofold movement is already evident in the realm of numbers, which we have considered as the prototype of a purely constructive order. The realm of real numbers could not have been constituted in the form it has taken in modern analysis if the whole number, conceived by the Pythagoreans as the primary principle of thought and being, had not pressed continuously beyond its own confines, if it had not been progressively amplified. The need for such an extension of the original numerical concept arose when this concept confronted questions that had not arisen within its own sphere but were propounded by the intuitive world, the world of magnitudes. It was primarily the problems connected with the measurement of distances that compelled number to burst its original limits and led to the discovery of the irrational. The irrational, as the term itself suggests, was at first regarded as something alien to number and to its inherent logos: as an *ἄλογον* and *ἄρρητον*. Yet it was through this opposition that number discovered its own intellectual power and inner wealth. The subsequent development consisted not merely in juxtaposing the world of magnitudes to the world of numbers as a new and different world but in transforming the step provoked by an outward impetus into an inwardly necessary, conceptual progress.

Modern analysis stands at the end of this logical process. As the foundation of his whole theory of irrational numbers, Dedekind states the proposition that it is possible, without any notion of measurable magnitudes, to create the pure, continuous realm of number by a finite system of simple logical operations—and that it is this intellectual instrument which first makes possible an intelligible notion of continuous space.[4] And this view also constitutes the principle and driving force of Cantor's theory of the continuum.[5] Thus it is characteristic of the numerical concept in the form which it takes in modern analysis that it preserves its absolute autonomy over against the spheres of concrete-intuitive being, with which number would seem to have been inwardly interwoven throughout its history. Henceforth, so far as its *foundation* is concerned, number ought to stand entirely by itself. And this same relationship prevails between constructive and empirical concept formation, between experience and mathematical-physical theory. Over and over again empirical intuition has proved to be the element that fertilizes the theory—but on the other hand the process of fertilization requires a vigorous germ in the theory itself. Contact with the world of intuition does not drive thought outside of itself but leads it deeper within itself, into its own "ground." Out of this ground it develops the new forms that can do justice to the complex structure of intuitive being. The history of exact science teaches us over and over that only such concepts as have thus grown out of the very source of thought have ultimately proved equal to experience. We may say, with an image borrowed from the language of chemistry, that sensory intuition acts as a catalyst for the development of scientific theory. It is indispensable for the process of exact concept formation—but it is no longer discoverable as an independent component in the product of this process or, one might say, in the logical substance of the exact concept. The farther this concept progresses, the more the sensuous-intuitive determinations from which it started, though not forgotten or annihilated, seem to be taken up into a new mode of formation. This change of form is not merely an outward relation between the otherwise unchanged elements of sensuous intuition; rather, it seizes the elements themselves at their root; it gives them a new meaning and in this meaning a new being.

This method of intellectual and symbolic formation may be illustrated

4. Richard Dedekind, *Stetigkeit und irrationale Zahlen* (2d ed. Braunschweig, 1892). Eng. trans. by Wooster Woodruff Beman, *Essays on the Theory of Numbers* (Chicago, 1909). Cf. the preface to Dedekind's *Was sind und was sollen die Zahlen?* p. xiii.

5. See Georg Cantor, *Grundlagen einer allgemeinen Mannigfaltigkeitslehre* (Leipzig, 1883), p. 29.

by an example in which the general trend is almost self-evident. Physics cannot build up its peculiar object world without the help of another constitutive concept beside that of number, namely the concept of space. These two elements can only be effective in their mutual permeation; and so close is their involvement that it even dominates the original discovery of the scientific concept of number. For the Pythagoreans the factor of number is still inseparable from that of space: the relations between numbers can be developed and described only if they are demonstrated to be spatial relations, relations between points. But significant and fruitful as this synthesis of space and number has proved for the history of mathematical and scientific thinking, it contains within it, from a purely logical point of view, the germ of a problem and dialectic which had burst forth as early as the paradoxes of Zeno.[6] For even if we assume that space, the form of outward intuition, inclines to the domination of the logos, the logos of space is still necessarily different from that of number. The two are very distinct in their logical structure. The manifold of spatial points and positions does not confront consciousness as a freely produced, synthetically constructed manifold. Here we cannot, as with number, start from the determination of a universal order, the order of sequence, and from it derive the whole wealth of special relations, in a strict and unbroken series of logical operations. Compared with this kind of derivation, space always seems to retain a character of the alogical: it cannot be exhausted through the pure activity of ordering, differentiating and relating. There remains an indissoluble residue: the specific form of space cannot be produced constructively, but can only be accepted as a mode of the given.

Here, then, we have a barrier which no "rationalization," however far it is carried, can surmount, but which thought must recognize at some point in its development. And the trend toward a thoroughgoing logicalization of mathematics, which dominates the modern development of analysis, seems not to have eliminated this barrier but to have brought it out all the more clearly. For Russell, who admits of no dividing line between the realm of numbers and that of purely logical form, whose whole effort is directed toward proving that the concept of number can be built up from purely logical constants, the problem of space creates a kind of

6. For the connection between Zeno's paradoxes and the problems of Pythagorean mathematics the reader is referred to my "Geschichte der antiken philosophie" in Dessoir's *Lehrbuch der Philosophie.*

logical hiatus. Russell, to be sure, also regards abstract geometry as a purely mathematical and hence strictly logical structure: its object differs from that of the pure theory of numbers only insofar as it investigates more complex forms of series, series of two or more dimensions. But this purely conceptual, hypothetical-deductive system of geometry contains no determination with regard to real, "actual space." This determination can be taken only from experience, so that the study of actual space becomes a branch of physics, of empirical science.[7] But precisely where the two spheres part, where pure thought seems to have come to the end of its resources, its meaning and goal are manifested in a new direction. For now the same fundamental relation as we have generally found to prevail between constructive and empirical manifolds, is confirmed in respect to the problem of space. The law of an empirical manifold cannot be established, it cannot be "found" by experience, except insofar as it has already been sought and in a certain sense anticipated theoretically. Without such an ideal anticipation the manifold of an empirical perception would never concentrate into a spatial form. The experience of the spatial is only possible if we ground this special experience in certain universal systems of order and measurement. We possess such systems of order and measurement, of diverse intellectual types, in the various kinds of projective, descriptive, and metric geometry. These systems contain essentially no statement of any kind about real things or facts; they state nothing but pure possibilities, an ideal readiness for the order of the factual. Experience as such contains no principle for the production of its possibilities; its role is limited to effecting a choice between them for application to each concrete individual case. Its actual achievement lies not in constitution, but in determination. The larger the sphere of possibilities, which thought has built up independently and spontaneously—the less thought is immured in itself—the more it stands open to experience and its determining function. Thus the hypothetical-deductive systems of geometry as such stand on the same logical plane as the pure numerical concepts. Experience as a constitutive factor enters into its foundations, its axioms, no more than it does, for example, into the realm of complex numbers.[8]

7. Cf. Russell, *Principles of Mathematics*, esp. pp. 372 ff.

8. In this conception of the relation between "geometry" and "experience," I stand, as far as I can see, closest to Max von Laue, among modern physicists. Cf. Vol. 2 of his *Die Relativitätstheorie* (Braunschweig, 1921), p. 29: "In 1864 Riemann took the step

If these systems independent of experience are to be made fruitful for experience; if a relation is to be created between the conceptual elements of geometry and the data provided by experience, a definite intellectual mediation is first of all required. For one series cannot be compared with the other or examined for similarity. Between empirical and ideal elements—as the discoverer of the ideal, Plato, clearly stated—there is no possible relation of similarity, of total or partial coincidence. Whatever community, whatever *κοινωνία* or *παρουσία* can be promoted between them, does not negate the fundamental character of "otherness," of *ἑτερότης* between them. Here similarity or congruence is replaced by the specific and new factor of "participation." This participation of the physical in the arithmetical and geometrical can be achieved and grounded in only one way: with certain physical "things" or "processes" we correlate certain mathematical concepts, but this correlation does not imply a relation of identity between them. Once the basic concepts and axioms of certain geometries are established, we may ask whether there are any elements of physical experience that are in agreement with these concepts and axioms. Thus, for example, a certain phenomenon, the phenomenon of the propagation of light, is used to provide a physical analogue to what is defined as a straight line in a certain hypothetical-deductive system of pure geometry. It is through such analogical relations that the concept of measurability first achieves an almost unrestricted meaning: it is through them that a determinate order of measure first follows from the ideal arithmetical order of number and the universal geometrical order of space. This order of measurement arises at exactly the point where, through the linking of geometrical concepts to physical experience, these concepts depart from the stage of abstract detachment and enter into a definite bond with the real, with the existence of physical phenomena. But this bond has no bearing on the validity of the con-

which later assumed fundamental importance for the universal theory of relativity: he established (in place of the simple Euclidean formula for distance $ds = \sqrt{dx_1^2 + dx_2^2 + dx_3^2}$) a homogeneous quadratic function of dx^i with any desired functions of x^i as coefficients $ds^2 = \sum_{ik} \gamma_{ik} dx^i dx^k$, as the square of the linear elements. This step may be called the universalized Pythagorean theorem. Every choice of the function γ_{ik} determines a particular kind of geometry. . . . At this point we should like only to state that nothing physical ever entered into this whole development of geometry from Euclid to Riemann. Deductions were drawn solely from certain axioms. These axioms themselves, to be sure, are not the only possible ones; the human mind can create others. But in establishing them it need borrow from experience no more than in creating the concept of complex number: geometries are therefore all a priori."

cepts and axioms as such; it applies only to the use we make of them in the determination of the elements of experience. We do not base the presuppositions and principles of Euclidean geometry on our experience of rigid bodies—rather, we make use of this experience in order to gain physical "correspondences" for the ideal statements of this geometry. According to the mode of these correspondences, according to our decision as to which bodies we wish to regard as rigid and which motions as rectilinear, our basic determination of measure changes, and with it the form of our geometry. In this sense—but only in this sense—every concrete geometry, every geometry that is characterized by a fixed determination of measurement, comprises certain physical presuppositions and postulates: but the fact that it can only be filled with empirical content through such postulates does not mean that it is logically grounded in this content. There must first be a universal order of number and a universal geometry, a science of possible spatial forms, before a determinate physical order of measurement can be constituted. It was in this sense that Leibniz characterized the methodological relation between the abstract and the concrete. Arguing against Locke, he wrote:

> Quoiqu'il soit vrai qu'en concevant le corps, on conçoit quelque chose de plus que l'espace, il ne s'en suit point qu'il y a deux étendues, celles de l'espace et celle du corps; car c'est comme lorsqu'en concevant plusieurs choses à la fois, on conçoit quelque chose de plus que le nombre, savoir *res numeratas,* et cependant il n'y a pas deux multitudes, l'une abstraite, savoir celle du nombre, l'autre concrète, savoir celle des choses nombrées. On peut dire de même qu'il ne faut point s'imaginer deux étendues, l'une abstraite de l'espace, l'autre concrète du corps; le concret n'étant tel que par l'abstrait.[9]

Here the strict idealist inference is drawn: the realm of the "idea" is recognized in its independence and original significance, but this meaning does not imply that pure space leads a separate existence side by side with the empirical, corporeal world.

Here again we have confirmation of the fact that the relation between the world of pure forms and the world of things never consists in correspondences between single forms and single things, but rather that only the two structures can be related to each other and measured by each

9. Leibniz, *Nouveaux essais sur l'entendement humain,* II, 4; *Philosophische Schriften,* 5, 115.

other as totalities. This, it is true, seems to make for an almost arbitrary freedom in the determination of the particular. Whether in order to give definite physical content to the concept of the "rectilinear" we connect it with the phenomenon of the propagation of light or whether we select a different analogy seems at first sight to be purely a question of choice, of free convention. But even this convention must be grounded in some way: it must, to speak in scholastic terms, have a *fundamentum in re*. However, the foundation cannot be disclosed in any particular thing, in an individual "this" and "that," but follows only from a synthesis of experience as a whole. We select those assumptions on the basis of which we can formulate a simple and systematically complete explanation of the natural phenomena. And since both the simplicity and the systematic completeness are always relative, there is always a possibility that we may arrive at another and more satisfactory result through a suitable variation of the original assumption. But this renunciation of absolute validity deprives the symbols of mathematics and exact science of none of their objective meaning. For they derive this meaning not from transcendent objects which stand behind them and which they copy, but from their achievement, their function of objectivization.

Even though this function may never achieve its end, may never arrive at a *non plus ultra,* its direction is established. The endlessness of the road does not negate the determinacy of the direction: for it is precisely through their relation to "infinitely distant" points that directions are defined. Here we see once more how all our knowledge of nature—insofar as it is knowledge—that is, an ideal goal and an ideal task—rests ultimately on an act of freedom, on a standpoint that reason appoints itself. Yet true freedom is not the opposite of obligation, but is its beginning and source. The first act: the choice of certain empirical elements which we take as correspondences to certain constructive forms, is free—but in the second and all subsequent acts we are bondsmen, unless thought, by a new act, suspends the whole fabric of inferences and begins with a new assumption. For of course it lies in the nature of the empirical manifold itself that it can never, in a strict sense, dissolve into a pure constructive manifold. It is never "constructed" to the end—but must always be conceived as indefinitely "constructible." Thus in its relation to the empirical datum the thread of thought never breaks off; but neither can it be spun to its end: such a conclusion would not mean the completion of the fabric, but its veritable destruction, because it would run counter to the meaning of experience as a progressive *process* of determination.[10]

The dimension of thought in which we are moving here may be elucidated and distinguished from other dimensions in reference to the natural space, the space of objective physical measurement. Space as such, conceived as a mere possibility of coexistence, has no definite and unambiguous form but stands equally open to the most diverse modes of formation. In his natural philosophy Plato called space the πρῶτον δεκτικόν; for him space is absolutely receptive and plastic, the raw material of all determination, which receives form and determinacy only through the legislation of the "idea." The philosophy of symbolic forms has given us a still broader notion of this inner plasticity that inheres in space, not limiting the sphere of the ideal to the realm of theoretical knowledge, but tracing the power and efficacy of the ideal back into other, deeper strata, particularly those of linguistic and mythological thinking, and finding that a peculiar mode of "spatiality" corresponds to each of these fields. The form of coexistence is always subject to a definite formative law, without which it could not be constituted, but in each instance the process of formation varies. At the present juncture we are concerned with understanding the transition leading from the empirical space of intuition to the conceptual space of theoretical physics. We recall that this empirical space of intuition proved to be shot through with symbolic elements, and that linguistic thinking in particular played an essential part in its whole formation and structure. With empirical space we have left the sphere of the mere datum far behind us; once language has coined its first indicative adverbs for the "here" and the "there," the near and the far, the process of construction has already begun. But in the progress to the space of abstract geometry and objective natural science, the formation assumes a wholly different character. Here, too, we start from certain elementary distinctions, which characterize space as a system of purely "topological" determinations. We apprehend relations of proximity between points of "apartness," the bisection of lines, the incidence of surfaces or segments of space. But from this manifold and complex fabric thought gradually detaches definite threads. It brings its own presuppositions and postulates to intuition, and so creates a new system of orientation. According to the modality of these presuppositions, a "projective" or a "metric" space develops from the original topological space. The form of metric space is contingent on an order of measurement which

10. For more detailed treatment, cf. my *Substanzbegriff und Funktionsbegriff*, pp. 410 ff.

appears freely chosen in the sense discussed above. We determine a body considered rigid, as being invariable in its measurements; we attribute the character of "straightness" to an empirically existing line. Through such postulations of measurement and straightness different "spaces" arise, each distinguished by a different structure.

Even the purely topological view comprises a theory of the combinations of spatial forms; here, too, a simply connected surface is distinguished from a multiple connected surface and very definite mathematical criteria are given for this difference. But our attention dwells exclusively on the relations of contiguity or of unbroken connection between the spatial forms, without forming a definite concept of their magnitude or shape. Size and shape become determinable only with the appearance of new postulations, new "hypotheses," whose special character determines each particular form of geometry.[11] If we now look back once again over the whole development of the motif of space, we perceive the whole extent of the oppositions that must be traversed. There is no single basic tendency or power of the human spirit that has not in some way participated in this vast process of formation and dominated it in certain phases. Sensation and intuition, feeling and fantasy, productive imagination and constructive conceptual thought are all equally at work—and in each instance the manner in which they interlock and condition one another creates a different form of space. But at the same time we find that with all its inner multiformity this process always preserves an identical direction and that in the process the sundering of the I and the world becomes increasingly clearer and more vivid in consciousness. The mythical consciousness of space still remains wholly interwoven with the sphere of subjective feeling. And yet even here the elementary contrasts in the primary life feeling give rise to definite oppositions in being, to a clash and separation of cosmic powers. Language carries this differentiation farther and gives it new depth: through its mediation mythical "expressive space" is transformed into "representative space." But

11. The most succinct and at the same time the most acute epistemological analysis of this relationship has, I believe, been given by Rudolf Carnap, *Der Raum. Ein Beitrag zur Wissenschaftslehre*, in *Kant-Studien, Ergänzungshefte* 56 (Berlin, 1922). Carnap distinguishes sharply between "formal" space (which is a pure structure of relation and order), "intuitive" space, and "physical" space. And he goes on to show how within each of these three spaces a definite distinction must be made between "topological," "projective," and "metric" space. Here I shall not enter into these distinctions—since in the framework of our problem we are concerned only with the principle of differentiation as such, and not with the concrete differences themselves—but refer the reader to Carnap's penetrating exposition.

only conceptual, geometrical, and physical thinking effects the last, decisive step. Here all the purely anthropomorphic components are suppressed more and more resolutely and replaced by strictly objective determinations which result from a universal method of counting and measuring. All elements arising from the sphere of feeling and even the images, the pure schemata of intuition, are progressively excluded. The step is taken from "expressive space" and "representative space" to a pure "significative space." [12] But this transition requires a number of other intermediary processes. The history of mathematics and the mathematical sciences shows that this process of transformation, though consistent, has been very gradual. Here we shall not follow the historical development but shall attempt, through the system of knowledge represented in modern physics, to single out the factors which throw light on the goal toward which this process tends and the methods it employs.