

Formal Logic

Critical Thinking

Recap: Formal Logic

- “If I win the lottery, then I am poor. I win the lottery. Hence, I am poor.”
- This argument has the following abstract structure or *form*: “If P then Q. P. Hence, Q”
- *Any* argument of the above form is valid, including “If flubbers are gook, then trugs are brig. Flubbers are gook. Hence, trugs are brig.”!
- Hence, we can look at the abstract form of an argument, and tell whether it is valid without even knowing what the argument is about!!
- Formal logic studies the validity of arguments by looking at the abstract form of arguments.

Using Formal Logic to Examine Validity of Arguments

- Formal logic works in 2 steps:
 - Step 1: Use certain symbols to express the abstract form of premises and conclusion.
 - Step 2: Use a certain procedure to figure out whether the conclusion follows from the premises based on their symbolized form alone.

Example

- “Either the housemaid or the butler killed Mr. X. However, if the housemaid would have done it, the alarm would have gone off, and the alarm did not go off. Therefore, the butler did it.”

Example

- First, let's clearly identify premises and conclusion:
 - Premise 1: Either the housemaid or the butler killed Mr. X.
 - Premise 2: If the housemaid would have killed Mr. X, the alarm would have gone off.
 - Premise 3: The alarm did not go off.
 - Conclusion: The butler did it.
- Note: Words like “however” or “therefore” are not part of the individual claims involved.

Step 1: Symbolization

- Use symbols to represent simple propositions:
 - H: The housemaid killed Mr. X
 - B: The butler killed Mr. X
 - A: The alarm went off
- Use further symbols to represent complex claims:
 - $H \vee B$: The housemaid or the butler killed Mr. X
 - $H \rightarrow A$: If the housemaid killed Mr. X, the alarm would go off
 - $\neg A$: The alarm did not go off

Step 2: Evaluation

- In a formal proof (or derivation), one uses basic inference rules to try and derive the conclusion from the premises:

| | | |
|-------|-------------------|---------|
| 1. | $H \vee B$ | A. |
| 2. | $H \rightarrow A$ | A. |
| 3. | $\neg A$ | A. |
| <hr/> | | |
| 4. | $\neg H$ | 2, 3 MT |
| 5. | B | 1, 4 DS |

Since every step along the way is an instance of an obviously valid inference, the conclusion does indeed follow from the premises. So, valid argument!

Warning!!!!

- Formal logic is a tool, but you have to become skilled in using this tool for it to be productive, rather than harmful.
- Here are some things to keep in mind:
 - Just as with mathematics, different systems have different applications. Or, even more general, different jobs call for different tools. So, know which logic system is appropriate to use when and where! And, know when not to use logic!
 - Most formal logic systems are systems to determine deductive validity, not inductive validity!
 - I'll indicate some other things along the way.

In Depth Coverage

Truth-Functional Logic

- Applies to reasoning dealing with compound sentences built from truth-functional operators like ‘and’, ‘or’, ‘not’, and ‘if ... then’.
- An operator is truth-functional in that the truth-value of a sentence like “P and Q” is a function of the truth-values of the sentences P and Q.

Symbolization

- “John is tall” or “There are 30 people in the classroom” are *simple (or atomic) claims*.
 - We use capital letters (or variables) to represent those: A, B, P, Q, etc.
- Claims containing any of the connectives such as ‘and’, ‘or’, ‘not’, and ‘if ... then’ are *compound claims*.
 - We use special logic symbols to symbolize those connectives:
 - ‘and’: \wedge or $\&$
 - ‘or’: \vee or $|$
 - ‘not’: \neg or \sim
 - ‘if ... then’: \rightarrow

Example

- “If Parsons signs the papers then Quincy will go to jail and Rachel will file an appeal”
- Does this mean: “If Parsons signs the papers,(!) then Quincy will go to jail and Rachel will file an appeal”
 - i.e. $P \rightarrow (Q \wedge R)$
- Or: “If Parsons signs the papers then Quincy will go to jail,(!) and Rachel will file an appeal”.
 - i.e. $(P \rightarrow Q) \wedge R$
- These two statements have different meanings!
 - English language is often ambiguous; logic is not

Truth-Tables

| P | $\neg P$ |
|---|----------|
| T | F |
| F | T |

| P | Q | $P \wedge Q$ |
|---|---|--------------|
| T | T | T |
| T | F | F |
| F | T | F |
| F | F | F |

| P | Q | $P \vee Q$ |
|---|---|------------|
| T | T | T |
| T | F | T |
| F | T | T |
| F | F | F |

Warning: Classical logic (such as truth-functional logic) assumes every claim is either true, or false. Claims cannot be both true and false, or neither true nor false. This seems like a reasonable assumption, but beware of the fact that most words in the English language are vague or 'fuzzy', meaning that expressions using those words may not have a crisp, clean, truth-value. E.g "Bram is tall" or "Bram's shirt is green". Or: "there are 9 planets in the solar system"! Some logic systems (such as fuzzy logic) try to address this.

The Material Conditional

- Let us define the binary truth-functional connective ' \rightarrow ' according to the truth-table below.
- The expression $P \rightarrow Q$ is called a *conditional*. In here, P is the *antecedent*, and Q the *consequent*.
- The conditional is intended to capture 'if ... then ...' statements.

| P | Q | $P \rightarrow Q$ |
|---|---|-------------------|
| T | T | T |
| T | F | F |
| F | T | T |
| F | F | T |

The Paradox of the Material Implication

- Consider the statement ‘If bananas are yellow, then elephants are big’.
 - Most people would consider this a false statement: the size of elephants has nothing to do with the color of bananas!
 - However, according to the truth-table, since both claims involved are true, the whole conditional should be true!
- Or, suppose I tell you that it is not true that ‘if there is a king in the hand, then there is an ace in the hand’.
What can be inferred?
 - Most people would say absolutely nothing. However, according to the truth-table, there is only one way for a conditional to be false, and that is when the antecedent is true, and the consequent false. Hence, formal logic says that there is a king in the hand, and not an ace!
- What is going on?

The Bad News

- There are many uses of the English ‘if ... then’ expression that don’t seem to be truth-functional: e.g. when we use an ‘if ... then’ to try and express a causal, temporal, or logical relationship, we can’t say anything about the truth-value of the whole ‘if ... then ...’ statement just because we know the truth-values of its parts.
- Remember, just because I mathematically define a certain system doesn’t mean that it applies to reality!

The Good News

- Fortunately, the validity of most arguments involving ‘if ... then ...’ claims merely depend on the observation that if the ‘if’ part is true, and the ‘then’ part false, then the whole ‘if ... then ...’ statement will be false, and indeed that should be the case no matter what!
- So, while the English ‘if ... then ...’ is often not truth-functional, in practice using the material conditional usually works out just fine.
- But be aware!!!

Necessary and Sufficient Conditions

- Conditionals can be used to express necessary and sufficient conditions:
- *Sufficient Condition*: Something (P) is a sufficient condition for something else (Q) iff P being the case guarantees Q being the case. Hence, if we know that P is true, we know Q is true: $P \rightarrow Q$
- *Necessary Condition*: Something (P) is a necessary condition for something else (Q) iff P being the case is required for Q being the case. Thus, while P may be true without Q being true, we do know that if Q is true, P is true: $Q \rightarrow P$

'If' vs 'Only if'

- Sufficient conditions are expressed in English using 'if', while necessary conditions are expressed using 'only if'.
- Thus:
 - 'If P then Q': $P \rightarrow Q$
 - 'P if Q': $Q \rightarrow P$
 - 'P only if Q': $P \rightarrow Q$
 - 'Only if P, Q': $Q \rightarrow P$

'Unless'

- A statement of the form 'P unless Q' usually means: 'P is the case as long as Q is not the case, but if Q is the case, then P is not the case'.
- However, the last part is not always intended. That is, sometimes we say 'P unless Q' to mean 'P is the case as long as Q is not the case. However, if Q is the case, then I don't know about P'.
 - Example: If I say: "You are not going to pass the final unless you study hard", I mean that if you don't study, you are not going to pass the final, but I don't mean that if you do study, you will pass the final!
- For this reason, we are going to translate 'P unless Q' with just $\neg Q \rightarrow P$ unless stated otherwise.

'If and only if' and the Material Biconditional

- A statement of the form 'P if and only if Q' (or 'P iff Q') is short for 'P if Q, and P only if Q'. Hence, we could translate this as $(P \rightarrow Q) \wedge (Q \rightarrow P)$. However, since this is a common expression, we define a new connective ' \leftrightarrow ':

| P | Q | $P \leftrightarrow Q$ |
|---|---|-----------------------|
| T | T | T |
| T | F | F |
| F | T | F |
| F | F | T |

Step 2: Evaluation

- Many different techniques have been developed:
 - Truth-Tables
 - Short Truth-Tables
 - Algebra's
 - Formal Proofs
 - Truth Trees

Truth-Table Method

- Besides *defining* the truth-conditions of our basic logical operators, truth-tables can also be used for *evaluating* the validity of arguments.
 - First, generate all possible truth-value combinations of the atomic claims involved in the argument.
 - Then, work out the truth-conditions of each of the statements involved
 - Finally, see if there is a row where all the premises are true and the conclusion is false.
 - If there is such a row, the argument is not valid.
 - If there is no such row, then the argument is valid.

Housemaid, Butler, and Alarm

Boole: Untitled 1

File Edit Table Window Help

\wedge \vee \neg \rightarrow \leftrightarrow \perp
 a b c d e f
 \forall \exists = \neq ()
 x y z u v w

Tet Small LeftOf SameCol Adjoin
 Cube Medium RightOf SameRow Betwe
 Dodec Large FrontOf Smaller SameSh
 SameSize BackOf Larger

Delete Column Verify Row
 Build Ref Cols Verify Table
 Fill Ref Cols Verify Assess

| Correct? | Complete? | Assessment (none given) | | | | | | |
|----------|-----------|-------------------------|-----|-----|------------|-------------------|----------|-----|
| | | =1= | =2= | =3= | (1) | (2) | (3) | (4) |
| | | H | B | A | $H \vee B$ | $H \rightarrow A$ | $\neg A$ | B |
| | | T | T | T | T | T | F | T |
| | | T | T | F | T | F | T | T |
| | | T | F | T | T | T | F | F |
| | | T | F | F | T | F | T | F |
| | | F | T | T | T | T | F | T |
| | | F | T | F | T | T | T | T |
| | | F | F | T | F | T | F | F |
| | | F | F | F | F | T | T | F |
| | | | | | (1) | (2) | (3) | (4) |

Short Truth Table Method

Remember: An argument is deductively valid if it is impossible for the premises to be true and the conclusion false. So, since we are interested in such a possibility, let's see what that would amount to. In other words, let's see if we can construct a counterexample to the argument:

| | |
|-------------------|--|
| $H \vee B$ | Try and make this T. (2) With B is F, H should be T |
| $H \rightarrow A$ | Try and make this T. (3) With H is T, A should be T |
| $\neg A$ | Try and make this T. (4) But with A is T, this is false! |
| <hr/> | |
| B | Try and make this F. (1) So B should be F |

We see that it is impossible to try and make the premises true and the conclusion false. So, the argument is valid!

De Morgan's Laws

| P | Q | $\neg(P \wedge Q)$ | | $\neg P \vee \neg Q$ | | | $\neg(P \vee Q)$ | | $\neg P \wedge \neg Q$ | | |
|---|---|--------------------|---|----------------------|---|---|------------------|---|------------------------|---|---|
| T | T | F | T | F | F | F | F | T | F | F | F |
| T | F | T | F | F | T | T | F | T | F | F | T |
| F | T | T | F | T | T | F | F | T | T | F | F |
| F | F | T | F | T | T | T | T | F | T | T | T |
| | | ↑ | | | ↑ | | ↑ | | | ↑ | |

So:

$$\neg(P \wedge Q) \Leftrightarrow \neg P \vee \neg Q$$

$$\neg(P \vee Q) \Leftrightarrow \neg P \wedge \neg Q$$

Some Logical Equivalences

- Double Negation:
 - $P \Leftrightarrow \neg\neg P$
- DeMorgan:
 - $\neg(P \wedge Q) \Leftrightarrow \neg P \vee \neg Q$
 - $\neg(P \vee Q) \Leftrightarrow \neg P \wedge \neg Q$
- Distribution:
 - $P \vee (Q \wedge R) \Leftrightarrow (P \vee Q) \wedge (P \vee R)$
 - $P \wedge (Q \vee R) \Leftrightarrow (P \wedge Q) \vee (P \wedge R)$
- Contradiction and Tautology:
 - $P \wedge \neg P \Leftrightarrow \perp$
 - $P \vee \neg P \Leftrightarrow \top$
 - $P \wedge \perp \Leftrightarrow \perp$
 - $P \vee \top \Leftrightarrow \top$
- Implication:
 - $P \rightarrow Q \Leftrightarrow \neg P \vee Q$
 - $\neg(P \rightarrow Q) \Leftrightarrow P \wedge \neg Q$
- Transposition (Contraposition):
 - $P \rightarrow Q \Leftrightarrow \neg Q \rightarrow \neg P$
- Equivalence:
 - $P \leftrightarrow Q \Leftrightarrow (P \rightarrow Q) \wedge (Q \rightarrow P)$

Algebra Method

- Capture all information of the premises into one big claims (probably one big conjunction).
- Then, use equivalences to transform (mainly: simplify!) to see if conclusion follows.

Housemaid, Butler, and Alarm

$(H \vee B) \wedge (H \rightarrow A) \wedge \neg A \Leftrightarrow$ (Implication)

$(H \vee B) \wedge (\neg H \vee A) \wedge \neg A \Leftrightarrow$ (Reduction)

$(H \vee B) \wedge \neg H \wedge \neg A \Leftrightarrow$ (Reduction)

$B \wedge \neg H \wedge \neg A$

Note: the result is not just 'B'

Formal Proof Method

- Use small and obviously valid inferences to work your way from the premises to the conclusion.
- If all steps are valid, then the original argument is valid as well.

Modus Ponens

$$\begin{array}{l} P \rightarrow Q \\ P \\ \hline Q \end{array}$$

| P | Q | $P \rightarrow Q$ | P | Q |
|---|---|-------------------|---|---|
| T | T | T | T | T |
| T | F | F | T | F |
| F | T | T | F | T |
| F | F | T | F | F |

←

We see that whenever the premises ($P \rightarrow Q$ and P) are true, the conclusion (Q) is also true. Hence, given the premises, the conclusion is necessarily true as well. So, this argument is deductively valid.

Some Valid Inferences

$$P \rightarrow Q$$
$$P$$
$$\hline Q$$

Modus Ponens

$$P \vee Q$$
$$\neg P$$
$$\hline Q$$

Disjunctive
Syllogism

$$P \rightarrow Q$$
$$\hline (P \wedge R) \rightarrow Q$$

Strengthening the
Antecedent

$$P \rightarrow Q$$
$$\neg Q$$
$$\hline \neg P$$

Modus Tollens

$$P \rightarrow Q$$
$$Q \rightarrow R$$
$$\hline P \rightarrow R$$

Hypothetical
Syllogism

$$P \rightarrow (Q \wedge R)$$
$$\hline P \rightarrow Q$$

Weakening the
Consequent

Some *Invalid* Inferences

$$P \rightarrow Q$$
$$Q$$

$$P$$

Affirming the
Consequent

$$P \rightarrow Q$$
$$\neg P$$

$$\neg Q$$

Denying the
Antecedent

Housemaid, Butler, and Alarm

| | | |
|-------|-------------------|---------------------------------|
| 1. | $H \vee B$ | A. |
| 2. | $H \rightarrow A$ | A. |
| 3. | $\neg A$ | A. |
| <hr/> | | |
| 4. | $\neg H$ | 2, 3 MT (Modus Tollens) |
| 5. | B | 1, 4 DS (Disjunctive Syllogism) |

Proof by Contradiction

- Proofs by contradiction follow the following pattern: 'Assume P to be the case. Then [after some work] I get some kind of impossibility or contradiction. Hence, contrary to my assumption, P cannot be the case.'
- This pattern of reasoning goes by many names:
 - *Proof by Contradiction*
 - *Indirect Proof*
 - *Reductio ad Absurdum*
 - *Reductio Proof*

Housemaid, Butler, and Alarm

| | | | |
|-------|--|-------------------|---------------------------------|
| 1. | | $H \vee B$ | A. |
| 2. | | $H \rightarrow A$ | A. |
| 3. | | $\neg A$ | A. |
| <hr/> | | | |
| 4. | | | H A. |
| <hr/> | | | |
| 5. | | | A 2, 4 MP (Modus Ponens) |
| 6. | | | \perp 3, 5 |
| 7. | | $\neg H$ | 4-6 IP (Indirect Proof) |
| 8. | | B | 1, 7 DS (Disjunctive Syllogism) |

Proof by Cases

- Proofs by Cases follow the following pattern:
'Either P is the case or Q is the case. However, if P is the case, then S is the case, and if Q is the case, then S is the case as well. Either way, S is therefore the case. Hence, S is the case.'
- This pattern of reasoning is called *Proof by Cases*
- Obviously, the above pattern can be generalized to disjunctions with any number of disjuncts.
- However, a very common form is to start with:
'Either P is the case or P is not the case'.

Housemaid, Butler, and Alarm

The screenshot shows the Fitch proof editor window titled "Fitch: Butler.prf". The interface includes a menu bar (File, Edit, Proof, Goal, Window, Help), a toolbar with navigation and proof symbols, and a main workspace. The workspace contains a proof tree on the left and a goal list on the right. The goal list shows the goal B is satisfied by the rule \vee Elim. A status bar at the bottom indicates "The goal checks out."

Proof Tree:

- $H \vee B$
- $H \rightarrow A$
- $\neg A$
- H
 - A
 - \perp
 - B
- B

Goal List:

- \rightarrow Elim
- \perp Intro
- \perp Elim
- \vee Elim

Goals:

- B

The goal checks out.

Formal Proofs vs Truth Tables

- Formal Proofs:
 - Correspond to step-by-step reasoning
 - Don't get so big fast
 - Require creativity
 - Can be used for propositional as well as predicate logic
 - Can demonstrate consequence (validity) only
- Truth Tables:
 - Correspond to exhausting possible models
 - Get big fast
 - Completely mechanical
 - Can be used for propositional logic only
 - Can demonstrate consequence (validity) as well as non-consequence (invalidity)

Demonstrating Invalidity through Counterexamples

Demonstrating Invalidity

- An argument is valid if the conclusion follows from the premises.
- In other words: *if* the premises are true, *then* the conclusion must be true as well.
- Another way of saying this is: an argument is valid if it is *impossible* for the conclusion to be false while the premises are true.
- Thus, to demonstrate *invalidity*, all we have to do is to demonstrate that it is *possible* for the conclusion to be false while the premises are true.

Counterexamples

- The easiest way to show that it is *possible* for the premises to be true and the conclusion to be false, is to come up with a scenario (or possible world) in which all premises *are* true and the conclusion false. This is called a counterexample world, or simply a counterexample to the argument

Example: Affirming the Consequent and Denying the Antecedent

$$P \rightarrow Q$$
$$Q$$

$$P$$

Affirming the
Consequent

$$P \rightarrow Q$$
$$\neg P$$

$$\neg Q$$

Denying the
Antecedent

Counterexample (for both):

P: Pat is pregnant

Q: Pat is a woman

Refutation by Formal Analogy I

- Example:
 - All politicians are well-intentioned. Some well-intentioned people are corrupt.
Therefore, some politicians are corrupt.
- It is hard to imagine a scenario/world where all politicians are well-intentioned, some well-intentioned people are corrupt, but no politicians are corrupt!
- What to do?

Refutation by Formal Analogy II

- Example:
 - All politicians are well-intentioned. Some well-intentioned people are corrupt. Therefore, some politicians are corrupt.
- Remember that an argument's validity is determined solely by its abstract form, and not by its actual content. This form is:
 - All P's are Q's. Some Q's are R's. Therefore, some P's are R's.
- Is this valid or invalid?

Refutation by Formal Analogy III

- Is this valid?
 - All P's are Q's. Some Q's are R's. Therefore, some P's are R's.
- Let's try and fill back in some things:
 - All children are people. Some people are over 65 years old. Therefore, some children are over 65 years old.
- Notice that the premises *are* true, and the conclusion is false. But then it is certainly *possible* for the premises to be true, and the conclusion false. So, this argument is invalid.

Refutation by Formal Analogy IV

- The original argument:
 - All politicians are well-intentioned. Some well-intentioned people are corrupt. Therefore, some politicians are corrupt.
- Has the exact same form as the new argument:
 - All children are people. Some people are over 65 years old. Therefore, some children are over 65 years old.
- Since the latter is clearly invalid, the former is therefore invalid as well.
- [Another refutation by formal analogy](#)