

# Paradoxes of Infinity

## Zeno's Paradox

Suppose that Achilles and the Tortoise are going to run a race, but since Achilles is so much faster than the Tortoise, the Tortoise gets a head start. Will Achilles win the race?

Zeno says no. His argument is as follows. In order to catch up with the Tortoise, Achilles first has to get to wherever the Tortoise is now. However, once Achilles gets there, the Tortoise will have moved forward, so Achilles is still behind, and hence the story repeats. But, this means that there is no end to the process of Achilles trying to catch up with the Tortoise. Hence, Achilles won't be able to catch up with the Tortoise.

What, if anything, is wrong with the logic of this argument?

### Response 1

"Well, obviously the argument is flawed, since Achilles will win the race. In fact, just do the race, and that will show the argument to be wrong."

The problem with this response is that it focuses on the conclusion of the argument. That is, the above response points out that the conclusion of the argument is false. However, that doesn't resolve the paradox. In fact, that's exactly what *makes* this a paradox: On the one hand, we have an argument that concludes that Achilles can't win, and on the other hand we know that the conclusion is totally wrong. Thus, the argument must be mistaken somewhere, but we don't know why. In order to resolve the paradox, we have to find a flaw in the reasoning.

### Response 2

"The argument only shows that there are an infinite number of time intervals that have to pass before Achilles can pass the Tortoise. However, that does not mean that the total amount of time that has to pass is infinite as well. In fact, you can sum up all these infinite time intervals, and end up with a perfectly finite amount of time."

This is a very popular response, but it misses the mark. That is, while it is indeed true that the sum of an infinite number of terms can be finite, the original argument merely argues that Achilles can't pass through an infinite number of points *without contemplating the time such an action might take*. Put differently, calculus tells us something about the possible *geometry* of the situation (maybe the finite interval between Achilles' starting point and the point where Achilles catches up with the Tortoise can be split into an infinite number of intervals), but it says nothing about its dynamics. Thus the paradox still stands: how can Achilles navigate such a geometry or, more to the point, how can Achilles ever move from point A to point B if that requires him to go through, and *finish* going through, an infinite sequence?).

### Response 3

“While the argument shows that there are an infinite number of points in time where Achilles is behind the Tortoise, this does not mean that Achilles is behind the Tortoise at every point in time. That is, since a strict subset of an infinite number of things can be infinite itself (viz. the even numbers of all numbers), it is possible to point to an infinite number of elements of an infinite set without having pointed to all members of that set.”

Again, this response only works by making Zeno’s argument into a straw man. Indeed, *if* Zeno’s argument would have said that there being an infinite number of points in time where Zeno is behind the Tortoise implies that Achilles is behind the Tortoise for all points in time, then this objection goes through for the reason provided: just because I can point to an infinite number of numbers that are even does not mean that all numbers are even, even if these two sets have the same cardinality! However, the original argument makes no such inference. The original argument merely claims that it is impossible for one to complete an infinite sequence.

### Response 4

“But it is perfectly possible to complete an infinite sequence! Simply do the first thing at  $t = \frac{1}{2}$ , the second at  $t = \frac{3}{4}$ , the third at  $t = \frac{7}{8}$ , etc. At  $t = 1$ , you’ll have done an infinite number of things. The infinite number of points in time that the argument points out Achilles is behind the Tortoise follows a similar progression. For example, assume that the Tortoise gets a 10 yard head start, that the Tortoise moves 1 yard a second, and Achilles 10 yards a second. Then all the argument does is point to an infinite series of points in time as follows:  $t = 1$ ,  $t = 1.1$ ,  $t = 1.11$ , etc. But, at  $t = 1 \frac{1}{9}$ , all these infinite points in time will have passed, and Achilles has caught up with the Tortoise”

In the literature, completing an infinite sequence of tasks is called a supertask, and arguments that supertasks can be performed usually take the above form. However, there is a real problem with this argument, because it assumes that time can pass through an infinite number of points, i.e. that the flow of time is a supertask. Thus, this argument for the possibility of performing supertasks is circular! In fact, I would argue that the sequence as described above can never happen: if time does through  $t = \frac{1}{2}$ ,  $\frac{3}{4}$ ,  $\frac{7}{8}$ , etc, then time simply cannot reach  $t = 1$ . So how is the flow of time at all possible? Well, see the next response.

### Response 5

"The argument assumes that space (and time) is continuous (dense), but if space (or time) is discrete, then the process of catching up does not go on indefinitely."

Personally, I think this is the best way to try and resolve Zeno’s paradox: I believe that Zeno’s argument shows that either space and time are discrete, or that our concept of motion is mistaken, or that our concepts of space and time are mistaken. At least I can understand the first option! What do you think?

## Other Paradoxes of Infinity

### Hilbert's Hotel

There is a hotel with an infinite number of rooms, and all rooms are occupied. Can a new guest be fitted in? The answer would seem to be no: all rooms are full. However, if you simply ask everyone to move to the next room, i.e. the occupant of room 1 goes to room 2, the occupant of room 2 goes to room 3, etc. then room 1 can be given to the new guest!

The paradox here is that while it seems that the new guest can't be fitted in, we have an argument that the guest can be fitted in. To resolve this paradox, we merely have to point out that it only seems as if the new guest can't be fitted in because we look at it from the original situation where the old guests occupy rooms 1 and up. And indeed, the new guest can't be fitted in with that constraint. However, this does not exclude other possible ways to try and make everyone fit, and the question was whether the guest can be fitted in, period.

Question: What if a bus shows up with an infinite, but still countable, number of new guests?

Question: What if an infinite, but still countable, number of buses show up with an infinite, but still countable, number of guests each?

### Thomson's Lamp

Suppose you turn on a lamp at  $t = 1/2$ , turn it off at  $t = 3/4$ , turn it on at  $7/8$ , etc. Is the lamp on or off at  $t = 1$ ?

This is more of a troubling question than a paradox, but there are two ways to make this into a paradox:

First, one could argue that while the lamp must be in some state or the other at  $t=1$ , there really is no reason for the lamp to be in any state as opposed to the other: there is nothing in the scenario that would make the on or off position somehow 'preferable'. Of course, there is an asymmetry between on and off in that we first turn the lamp on rather than off, but this still seems a far cry from providing any metaphysical justification for as to why the lamp should be on rather than off at  $t=1$ .

However, stating a lack of reason for something to happen is not the same as providing a reason as to why something is plainly impossible. Thus Thomson (who came up with this supertask scenario) tried the following argument: the lamp must either be on or off at  $t=1$ . However, the lamp cannot be on, since every time it was turned on, it was turned off again immediately afterward. Similarly, the lamp cannot be off either, since everytime the lamp was turned off, the lamp was turned on again immediately afterward. Thomson used this argument to argue against the possibility of performing supertasks in general.

But there is a problem with Thomson's argument. As Benacerraf points out, the lamp being turned off (on) everytime it is on (off) only holds true for  $t < 1$ . But, this does not rule out the lamp being on (off) at  $t=1$ . Thus, the situation is that while we know for any  $t < 1$  whether the lamp is on or off, we simply don't know this for  $t=1$ : it may be on, and it may be off, and in fact there may be no good reason for either one state, but contrary to Thomson's claim, there is simply no contradiction in it being on (or off) at  $t=1$ !

Personally, I think Thomson's argument is indeed flawed for the reasons as indicated. However, I also think that in the scenario,  $t=1$  simply cannot be reached, if time has to go through points  $t=1/2, 3/4$ , etc. That is, Thomson argued as follows: "Assume this supertask is possible. Then the lamp is either on or off at  $t=1$ . But it can't be either, because <see above>. Contradiction. So, this supertask is impossible". I would argue this way: "Assume that the scenario is possible. Immediate contradiction in the assumption that time is continuous and also get to  $t=1$ ! So, the scenario is not possible."

### **Ross-Littlewood Paradox**

Suppose you have an infinite number of balls. At  $t=1/2$  you put two balls into a bin, and then take one out again. At  $3/4$  you do the same. Etc. It would seem that at  $t=1$ , there are an infinite number of balls in the bin. However, suppose that the balls were numbered 1,2, etc, and suppose that the balls you put in at  $t=1/2$  were 1 and 2, and that you took out 1, that at  $t=3/4$  you put in 3 and 4, and take out 2, etc. Then, at  $t=1$ , you cannot have any balls in the bin, since any ball left in the bin at  $t=1$  would have a number  $k$ , and would have been taken out at  $t = 1 - 2^{-k}$ .

Some authors argue that this is not a paradox because the number of balls left in the bin simply depends on how you put them in and how you take them out. In the last scenario, you would indeed have no balls left in the bin at  $t=1$ , but if you would take out ball 2 after putting in 1 and 2, and then take out 4 after putting in 3 and 4, you would have an infinite number of balls left in the bin.

Question: Following this kind of logic, how can I change the scenario and have exactly 1 ball left in the bin at  $t=1$ ? And how can I end up with exactly 17?

Still, it would seem that the paradox stands as follows: since at every time point you effectively add 1 ball to the bin, the number of balls in the bin can only increase, not decrease. Indeed, after an infinite number of points in time, the bin must have an infinite number of balls. Thus we have a genuine paradox. One can also point to the fact that the two scenarios of putting in and taking out balls are isomorph, and effectively equivalent as far as the number of balls at any point in time goes. So, the fact that we reach different outcomes for the 2 scenarios is a paradox.

I agree that this is a real paradox. I also think the solution is quite simple: once again, I would deny the possibility of the very set-up of this experiment: if time is continuous, then we simply can't reach  $t=1$ !