Sets

A *set* is a collection of objects. These objects can be anything, even sets themselves. An object x that is in a set S is called an *element* of that set. We write $x \in S$.

Some special sets:

$\emptyset$: The *empty set*; the set containing no objects at all

N: The set of all *natural numbers* 1, 2, 3, … (sometimes, we include 0 as a natural number)

Z: the set of all *integers* …, -3, -2, -1, 0, 1, 2, 3, …

Q: the set of all *rational numbers* (numbers that can be written as a ratio p/q with p and q integers)

R: the set of *real numbers* (points on the continuous number line; comprises the rational as well as irrational numbers)

When all elements of a set A are elements of set B as well, we say that A is a *subset* of B. We write this as $A \subseteq B$.

When A is a subset of B, and there are elements in B that are not in A, we say that A is a *strict subset* of B. We write this as $A \subset B$.

When two sets A and B have exactly the same elements, then the two sets are the same. We write $A = B$.

Some Theorems:

For any sets A, B, and C:

$A \subseteq A$

$A = B$ if and only if $A \subseteq B$ and $B \subseteq A$

If $A \subseteq B$ and $B \subseteq C$ then $A \subseteq C$

Formalizations in first-order logic:

$\forall x \, \forall y \, (x = y \leftrightarrow \forall z \, (z \in x \leftrightarrow z \in y))$ Axiom of Extensionality

$\forall x \, \forall y \, (x \subseteq y \leftrightarrow \forall z \, (z \in x \rightarrow z \in y))$ Definition subset

Operations on Sets

With A and B sets, the following are sets as well:

The *union* $A \cup B$, which is the set of all objects that are in A or in B (or both)

The *intersection* $A \cap B$, which is the set of all objects that are in A as well as B

The *difference* $A \setminus B$, which is the set of all objects that are in A, but not in B

The *powerset* $P(A)$ which is the set of all subsets of A.

The *Cartesian product* $A \times B$, which is the set of all ordered pairs (2-tuples) $\langle a, b \rangle$ where $a \in A$ and $b \in B$.
This generalizes to any number of sets, i.e. $A_1 \times A_2 \times \ldots \times A_n$ is the set of all n-tuples $\langle a_1, a_2, \ldots, a_n \rangle$ where $a_1 \in A_1$, $a_2 \in A_2$, … , and $a_n \in A_n$.

Some theorems:

$P(\emptyset) = \{\emptyset\}$

For any sets A, B, and C:

$A \cup B = B \cup A$	Commutation $\cup$
$A \cap B = B \cap A$	Commutation $\cap$
$A \cup (B \cup C) = A \cup (B \cup C)$	Association $\cup$
$A \cap (B \cap C) = A \cap (B \cap C)$	Association $\cap$
$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$	Distribution $\cup$ over $\cap$
$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$	Distribution $\cap$ over $\cup$

Formalization in FOL:

$\forall x \forall y (x \in pow(y) \leftrightarrow x \subseteq y)$	Definition powerset
$\forall x \neg x \in e$	Definition empty set
$\forall x \forall y \forall z (z \in union(x,y) \leftrightarrow (z \in x \vee z \in y))$	Definition union
$\forall x \forall y \forall z (z \in int(x,y) \leftrightarrow (z \in x \wedge z \in y))$	Definition intersection

Exercise: formalize the difference operation (use dif(x,y))

Relations

A *relation over* sets A_1, A_2, … , and A_n is a subset of $A_1 \times A_2 \times \ldots \times A_n$

A relation over two sets, A and B, is called a binary relation. For a binary relation R we often write aRb instead of $<a,b> \in R$. A binary relation R over A and B is:

Left-Total iff for all $a \in A$ there exists at least one $b \in B$ such that aRb

Right-Total iff for all $b \in B$ there exists at least one $a \in A$ such that aRb

Right-Unique iff for all $a \in A$ there exists at most one $b \in B$ such that aRb

Left-Unique iff for all $b \in B$ there exists at most one $a \in A$ such that aRb

A binary relation that is both left-unique and right-unique is a *one-to-one relation*

A binary relation that is both left-total and right-total is a *correspondence*

R is an (*endo-)relation on* A if R is a binary relation over A and A. A relation R on A is:

Reflexive iff aRa for all $a \in A$

Non-Reflexive iff not reflexive

Irreflexive iff not aRa for all $a \in A$

Symmetric iff for all $a \in A$ and $b \in A$: if aRb then bRa

Non-Symmetric iff not symmetric

Asymmetric iff for all $a \in A$ and $b \in A$: if aRb then not bRa

Anti-symmetric iff for all $a \in A$ and $b \in A$ where $a \neq b$: if aRb then not bRa
(or, what is the same thing: if aRb and bRa then $a = b$

Transitive iff for all $a \in A$, $b \in A$, and $c \in A$: if aRb and bRc then aRc

Non-Transitive iff not transitive

Antitransitive iff for all $a \in A$, $b \in A$, and $c \in A$: if aRb and bRc then not aRc

Theorem: A relation R is asymmetric if and only if R is anti-symmetric and irreflexive

Exercise: Prove this (using non-formal, but still proper mathematical proof)

Formalization in FOL:

$\forall x\ \exists y\ R(x,y)$	R is Left-total
$\forall y\ \exists x\ R(x,y)$	R is Right-total
$\forall x\ \forall y\ \forall z\ ((R(x,y) \wedge R(x,z)) \rightarrow y = z)$	R is Right-unique
$\forall x\ \forall y\ \forall z\ ((R(y,x) \wedge R(z,x)) \rightarrow y = z)$	R is Left-unique
$\forall x\ R(x,x)$	R is reflexive
$\forall x\ \neg R(x,x)$	R is irreflexive
$\forall x\ \forall y\ (R(x,y) \rightarrow R(y,x))$	R is symmetric
$\forall x\ \forall y\ (R(x,y) \rightarrow \neg R(y,x))$	R is asymmetric
$\forall x\ \forall y\ ((R(x,y) \wedge R(y,x)) \rightarrow x = y)$	R is anti-symmetric
$\forall x\ \forall y\ \forall z\ ((R(x,y) \wedge R(y,z)) \rightarrow R(x,z))$	R is transitive
$\forall x\ \forall y\ \forall z\ ((R(x,y) \wedge R(y,z)) \rightarrow \neg R(x,z))$	R is intransitive
$\forall x\ \forall y\ (R(x,y) \vee R(y,x))$	R is total
$\forall x\ \forall y\ (\neg x = y \rightarrow (R(x,y) \vee R(y,x))$	R is connex

Some special kinds of relations:

A reflexive, symmetric, and transitive relation is an *equivalence relation*

A reflexive, anti-symmetric, and transitive relation is a *partial order*

A total, anti-symmetric, and transitive relation is a *total (or linear) order*

An irreflexive, anti-symmetric, and transitive relation is a *strict partial order*

A connex, irreflexive, anti-symmetric, and transitive relation is a *strict total order*

Examples:

$\subseteq$ on sets is a partial (but non-strict) order, $\subset$ is partial and strict
$\leq$ on numbers is total (but non-strict) order, $<$ is a strict total order

Exercise: Show that all total orders are partial orders.

Functions

A function $f : A \rightarrow B$ is a binary relation over A and B that is right-unique.

(For this reason, a binary relation that is right-unique is often called functional).

A is called the *domain*, and B the *co-domain*, of the function f (note: many mathematicians use the word domain to refer to the set of objects for which a function-value is defined. We can call that the *domain of definition*. In the context of our course, however, it is more useful to regard the domain as the *domain (or universe) of discourse*.)

If there is an object b such that $<a, b> \in f$, then $f(a)$ is defined as b, i.e. $f(a) = b$

If there is no object b such that $<a, b> \in f$, then $f(a)$ is undefined

Frequently, the domain A is a Cartesian product $A_1 \times A_2 \times \ldots \times A_n$. Accordingly, we will write $f : A_1 \times A_2 \times \ldots \times A_n \rightarrow B$ and $f(a_1, a_2, \ldots , a_n)$ instead of $f(<a_1, a_2, \ldots , a_n>)$

The *range* of a function $f : A \rightarrow B$ is the set of all $b \in B$ for which there exists some $a \in A$ such that $f(a) = b$

A function $f : A \rightarrow B$ is:

Total iff $f(a)$ is defined for all $a \in A$

(Thus, a total function is a right-unique, left-total binary relation)

Partial iff f is not total

(Thus, a total function is a right-unique, but not left-total binary relation)

Surjective (or *onto*) iff for all $b \in B$, there exists some $a \in A$ such that $f(a) = b$

(Thus, a surjective function is a right-unique, right-total binary relation. A right-total binary relation is sometimes called a surjective relation. Also, note that a function is surjective iff its range equals its co-domain)

Injective (or *one-to-one*) iff there do not exist $a_1 \in A$ and $a_2 \in A$ such that $a_1 \neq a_2$ and $f(a_1) = f(a_2)$

(Thus, an injective function is a right-unique, left-unique binary relation. A left-unique binary relation is sometimes called an injective relation.)

Bijective iff f is surjective and injective

(So this is a right-unique, right-total, left-unique binary relation.)

A *one–to-one correspondence between* A and B iff f is total and bijective

> (So this is a binary relation that has all 4 interesting properties: right-unique, right-total, left-unique, and left-total. Conceptually, this means that every element from A corresponds to exactly one element in B, and vice versa. Indeed, the name makes sense: remember that a binary relation that is left-total and right-total is a correspondence, i.e. this is where there is at least one element from B is associated for every element from A, and vice versa. Accordingly, a functional correspondence would be a function (right-unique) that is total (left-total) and onto (right-total). Add one-to-one (left-unique), and you not only have a functional one-to-one correspondence, but also a binary relation that is right-unique, right-total, left-unique, and left-total.
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> Somewhat more confusingly, a right-unique, right-total, left-unique, left-total binary relation is sometimes called a bijective relation. This seems confusing, because if for functions (right-unique), bijective simply means surjective (right-total) and injective (left-unique), where does the left-total property suddenly come from? Well, when mathematicians talk about functions, they often suppose them to be total (left-total!), unless explicitly specified to be partial. Indeed, when mathematicians talk about a bijective function, they often mean a one-to-one correspondence. Likewise, mathematicians often refer to a one-to-one correspondence as simply a correspondence.)

For a one-to-one function function $f : A \rightarrow B$, we can define its *inverse function* $f^{-1} : B \rightarrow A$ as follows:

$f^{-1}(b) = a$ if $f(a) = b$
$f^{-1}(b)$ = undefined otherwise

Note that since f is one-to-one, there can at most be one a such that f(a) = b, so this function is well-defined.

For any two functions $f : A \rightarrow B$ and $g : B \rightarrow C$ we can define the *composite function* (or their *composition)* $gf : A \rightarrow C$ as follows:

$gf(a) = g(f(a))$ if $f(a)$ is defined
$gf(a)$ = undefined otherwise

Exercise: Show that if f is a one-to-one function, and f^{-1}its inverse:

> f is total if and only if $f^{-1}$is onto
>
> f is onto if and only if $f^{-1}$is total