Predicate Logic

From Propositional Logic to Predicate Logic

- Last week, we dealt with propositional (or truth-functional, or sentential) logic: the logic of truth-functional statements.
- Today, we are going to deal with predicate (or quantificational, or first-order) logic.
- Quantificational logic is an extension of, and thus builds on, truth-functional logic.

The Language of Predicate Logic

Individual Constants

- An individual constant is a name for an object.
- Examples: john, marie, a, b
- Each name is assumed to refer to a unique individual, i.e. we will not have two objects with the same name.
- However, each individual object may have more than one name.

Predicates

- Predicates are used to express properties of objects or relations between objects.
- Examples: Tall, Cube, LeftOf, =
- *Arity*: the number of arguments of a predicate (E.g. Tall: 1, LeftOf: 2)

Interpreted and Uninterpreted Predicates

- Just as ‘P’ can be used to denote any statement in propositional logic, a predicate like ‘LeftOf’ is left ‘uninterpreted’ in predicate logic. Thus, a statement like LeftOf(a,a) can be true in predicate logic.
- The predicate ‘=‘ is an exception: it will automatically be interpreted as the identity predicate.

Quantification: ‘All’ and ‘Some’

- In quantificational logic, there are two quantifiers: ‘all’ and ‘some’.
- Here are some examples:
  - $\forall x$ Mortal(x) ‘All things are mortal’
  - $\exists x$ Mortal(x) ‘Some things are mortal’
  - $\forall x$ (Human(x) $\rightarrow$ Mortal(x)) ‘Every human is mortal’
  - $\exists x$ (Human(x) $\wedge$ $\neg$Mortal(x)) ‘Some human is not mortal’

Universe of Discourse (or Domain)

- When we say ‘all’ or ‘some’, we mean ‘all’ or ‘some’ of a group of objects we have in mind.
- This group of objects is the Universe of Discourse or Domain

Symbolization

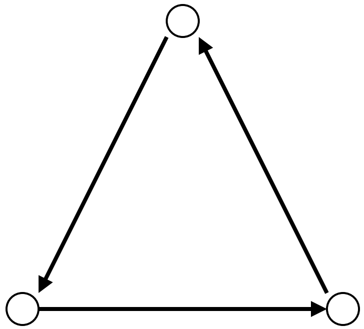
The Four Aristotelian Forms

- "All P's are Q's"
  - $\forall x\ (P(x) \rightarrow Q(x))$
- "Some P's are Q's"
  - $\exists x\ (P(x) \land Q(x))$
- "No P's are Q's"
  - $\forall x\ (P(x) \rightarrow \neg Q(x))$
- "Some P's are not Q's"
  - $\exists x\ (P(x) \land \neg Q(x))$

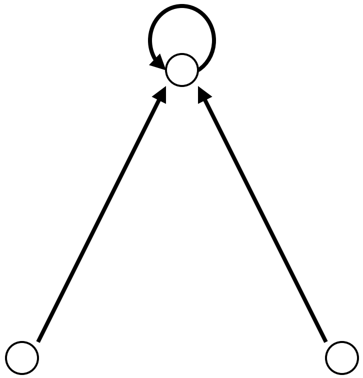
Translating Complex Phrases

- When translating (symbolizing) statements in FOL, clearly separate between the subject term (that about which you say something), and the predicate term (that what you say about those things)
- "Some blue cubes are big"
  - I am saying something about some blue cubes ...
    - $\exists x \; ((Cube(x) \land Blue(x)) \land \ldots$
  - and that is that they are big.
    - $\exists x \; ((Cube(x) \land Blue(x)) \land Big(x))$
- "No cubes are both blue and big"
  - I am saying something about all cubes …
    - $\forall x \; ((Cube(x) \rightarrow \ldots$
  - and that is that they are not both blue and big.
    - $\forall x \; ((Cube(x) \rightarrow \neg(Blue(x) \land Big(x)))$

Swapping Mixed Quantifiers: Order Matters



$\forall x \, \exists y$ Likes(x,y)
"Everything likes something (possibly itself)"



$\exists y \, \forall x$ Likes(x,y)
"Something is liked by everything (including itself)"

Expressing Number of Objects

- How do we express that there are (at least) two cubes?
- Note that $\exists x \exists y (Cube(x) \land Cube(y))$ doesn't work: this will be true in a world with 1 object (just pick that object for both x and y!)
- So, we have to make sure that x and y are different objects: $\exists x \exists y (x \neq y \land Cube(x) \land Cube(y))$

‘Exactly One’

- How can we say that “There is exactly one cube”?
- Saying that there is exactly one cube is saying two things at once:
  - There is at least one cube: $\exists x Cube(x)$
  - There is at most one cube: $\neg\exists x\exists y(Cube(x)\wedge Cube(y) \wedge x\neq y)$
  - Thus: $\exists x Cube(x) \wedge \neg\exists x\exists y(Cube(x)\wedge Cube(y)\wedge x\neq y)$
- Alternatively (and simpler):
  - $\exists x(Cube(x) \wedge \neg\exists y(Cube(y) \wedge x\neq y))$
  - $\exists x(Cube(x) \wedge \forall y(Cube(y) \rightarrow x=y))$
  - $\exists x\ \forall y(Cube(y) \leftrightarrow x=y))$

‘Exactly Two’

- How do we say “There are exactly two cubes”?
- Similar set-up:
  - $\exists x\ \exists y(\text{Cube}(x) \land \text{Cube}(y) \land x \neq y \land \neg\exists z(\text{Cube}(z) \land z \neq x \land z \neq y))$ or:
  - $\exists x\ \exists y(\text{Cube}(x) \land \text{Cube}(y) \land x \neq y \land \forall z(\text{Cube}(z) \rightarrow (z=x \lor z=y)))$ or:
  - $\exists x\ \exists y(x \neq y \land \forall z(\text{Cube}(z) \leftrightarrow (z=x \lor z=y)))$

‘Exactly n’

- Following previous set-up:
  - $\exists x_1 \exists x_2 \ldots \exists x_n(x_1 \neq x_2 \land \ldots \land x_1 \neq x_n \land x_2 \neq x_3 \ldots x_2 \neq x_n \land \ldots \land x_{n-1} \neq x_n \land \forall z(Cube(z) \leftrightarrow (z=x_1 \lor z=x_2 \lor \ldots \lor z=x_n)))$
- Alternatively, conjunct ‘at least n cubes’ with ‘at most n cubes’.
  - ‘At most n cubes’: $\exists x_1 \exists x_2 \ldots \exists x_n \forall z(Cube(z) \rightarrow (z=x_1 \lor z=x_2 \lor \ldots \lor z=x_n)))$
  - ‘At least n cubes’ (= ‘not at most n-1 cubes’): $\neg\exists x_1 \exists x_2 \ldots \exists x_{n-1} \forall z(Cube(z) \rightarrow (z=x_1 \lor z=x_2 \lor \ldots \lor z=x_{n-1})))$ (note: you make the Assumption of Existental Import here, i.e. that there is exists at least one object)

The Logic of Quantifiers

Quantifiers and Truth-Functional Logic

- Quantificational logic is an extension of truth-functional logic, so truth-functional relationships still exist in quantificational logic.
- To see if any truth-functional relationships hold when dealing with quantificational sentences, it is helpful to consider the *truth-functional form* of those sentences. To find the truth-functional form, simply substitute P, Q, etc for sentences.
- Example: $\forall x$ Cube(x) $\vee \neg\forall x$ Cube(x) has the truth-functional form $P \vee \neg P$, and therefore is a truth-functionally necessary true statement.

FO Necessities

- While $\forall x\ (Cube(x) \vee \neg Cube(x))$ is a logically necessary true statement, this is not so in virtue of truth-functional logic, since it has the truth-functional form P.
- The above statement is a necessarily true statement in virtue of truth-functional properties as well as quantificational properties (and identity).
- Thus, the above statement is said to be a quantificationally necessary true statement, or a first-order (FO) necessary true statement.
- For some strange reason, FO necessary true statements are also called FO valid statements.

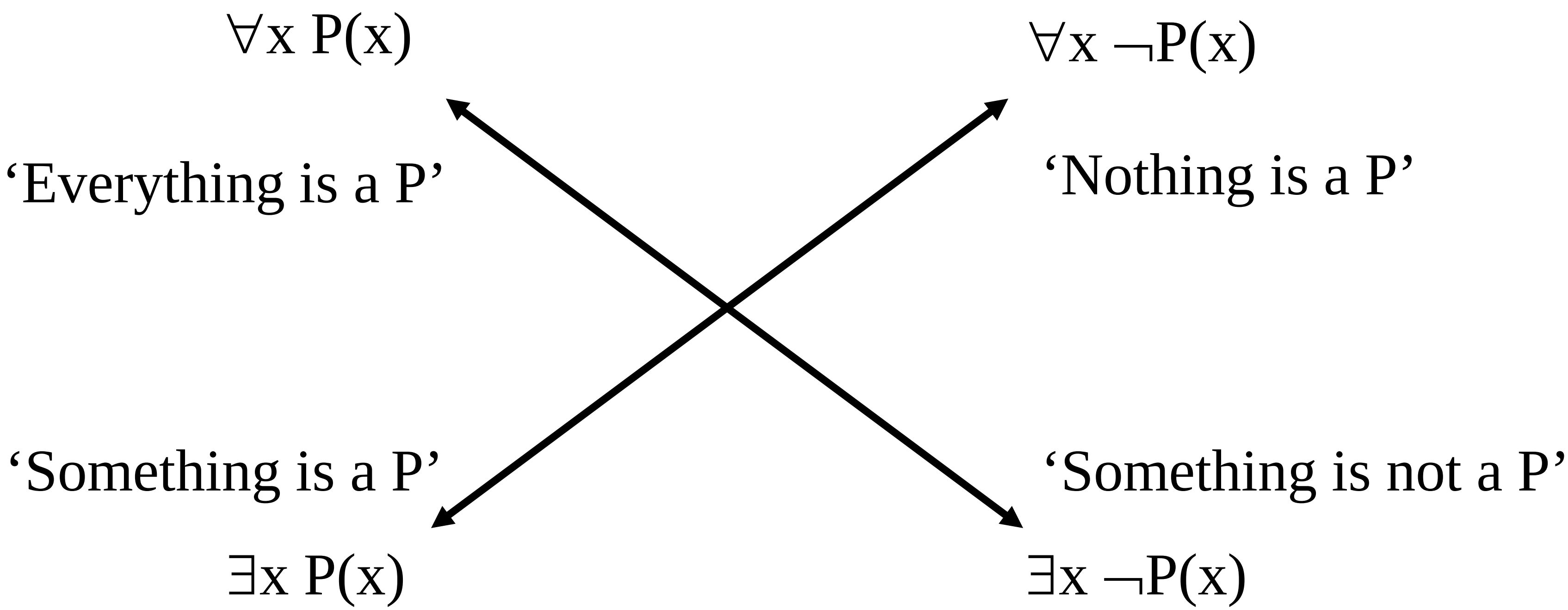
FO Consequence, Equivalence, Consistency, etc.

- The notions of FO consequence, equivalence, consistency, etc. can be similarly defined:
  - A statement ψ is a FO consequence of a set of statements Γ iff ψ is a logical consequence of Γ in virtue of truth-functional properties, quantificational properties, and identity.
  - Two statements φ and ψ are FO equivalent iff φ and ψ are logically equivalent in virtue of truth-functional properties, quantificational properties, and identity.
  - A set of statements Γ is FO consistent iff Γ is logically consistent in virtue of truth-functional properties, quantificational properties, and identity.
  - Etc.

Truth-Functional, First-Order, and Logical Consequence

- FO consequence sits between truth-functional consequence and logical consequence:
  - Remember we wrote $\Gamma \Rightarrow_{TF} \psi$ to indicate that ψ is a truth-functional consequence of Γ.
  - Let us now write $\Gamma \Rightarrow_{FO} \psi$ to indicate that ψ is a FO consequence of Γ. Then:
  - For any Γ and ψ, if $\Gamma \Rightarrow_{TF} \psi$, then $\Gamma \Rightarrow_{FO} \psi$, but not vice versa. E.g: $\{\neg\forall x\ Cube(x)\} \Rightarrow_{FO} \exists x \neg Cube(x)$, but not $\{\neg\forall x\ Cube(x)\} \Rightarrow_{TF} \exists x\ \neg Cube(x)$.
  - For any Γ and ψ, if $\Gamma \Rightarrow_{FO} \psi$, then $\Gamma \Rightarrow \psi$, but not vice versa. Example: $\{LeftOf(a,b)\} \Rightarrow RightOf(b,a)$, but not $\{LeftOf(a,b)\} \Rightarrow_{FO} RightOf(b,a)$.

The Boolean Square of Opposition

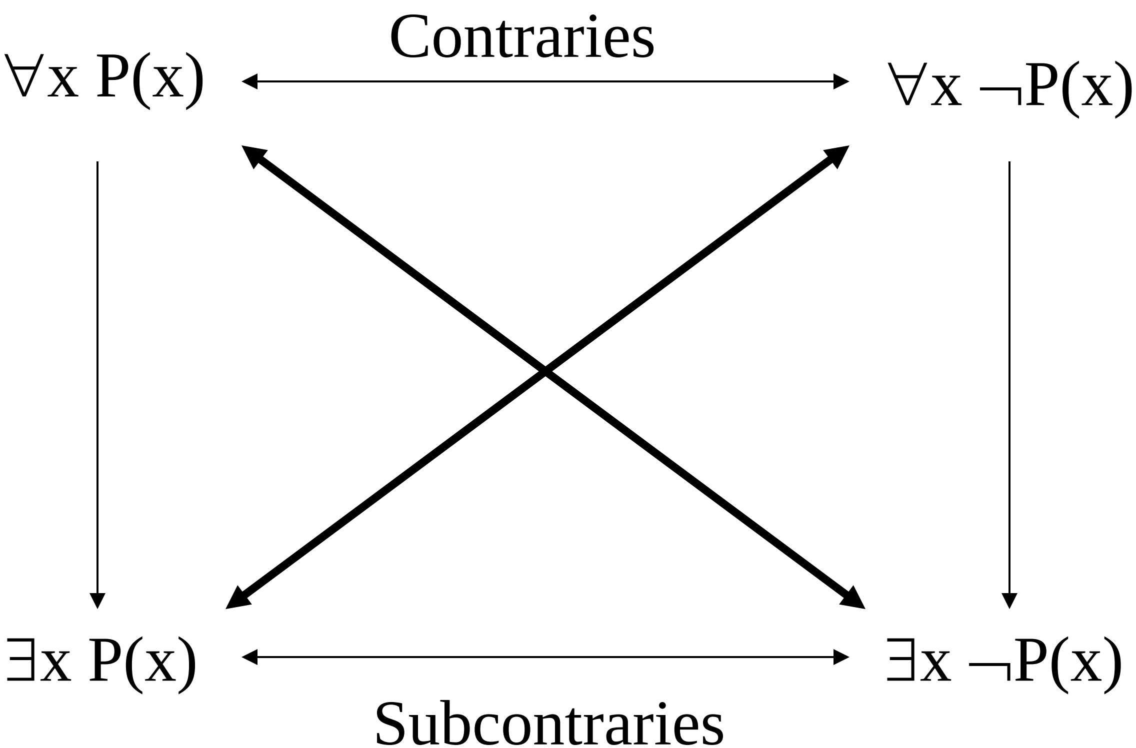


⟷ : Contradictories

The Assumption of Existential Import

- The Assumption of Existential Import is the assumption that the world in which we evaluate is not empty, i.e. that at least one thing exists.
- Under this assumption, $\exists x\ P(x)$ is true if $\forall x\ P(x)$ is true. Without the assumption, however, it’s not: if the world in which we evaluate is empty, then $\exists x\ P(x)$ is false, even though $\forall x\ P(x)$ is (vacuously) true.
- In first-order logic, we make the assumption of existential import. Thus, $\exists x\ P(x)$ is considered a FO consequence of $\forall x\ P(x)$, even though logically it is not.

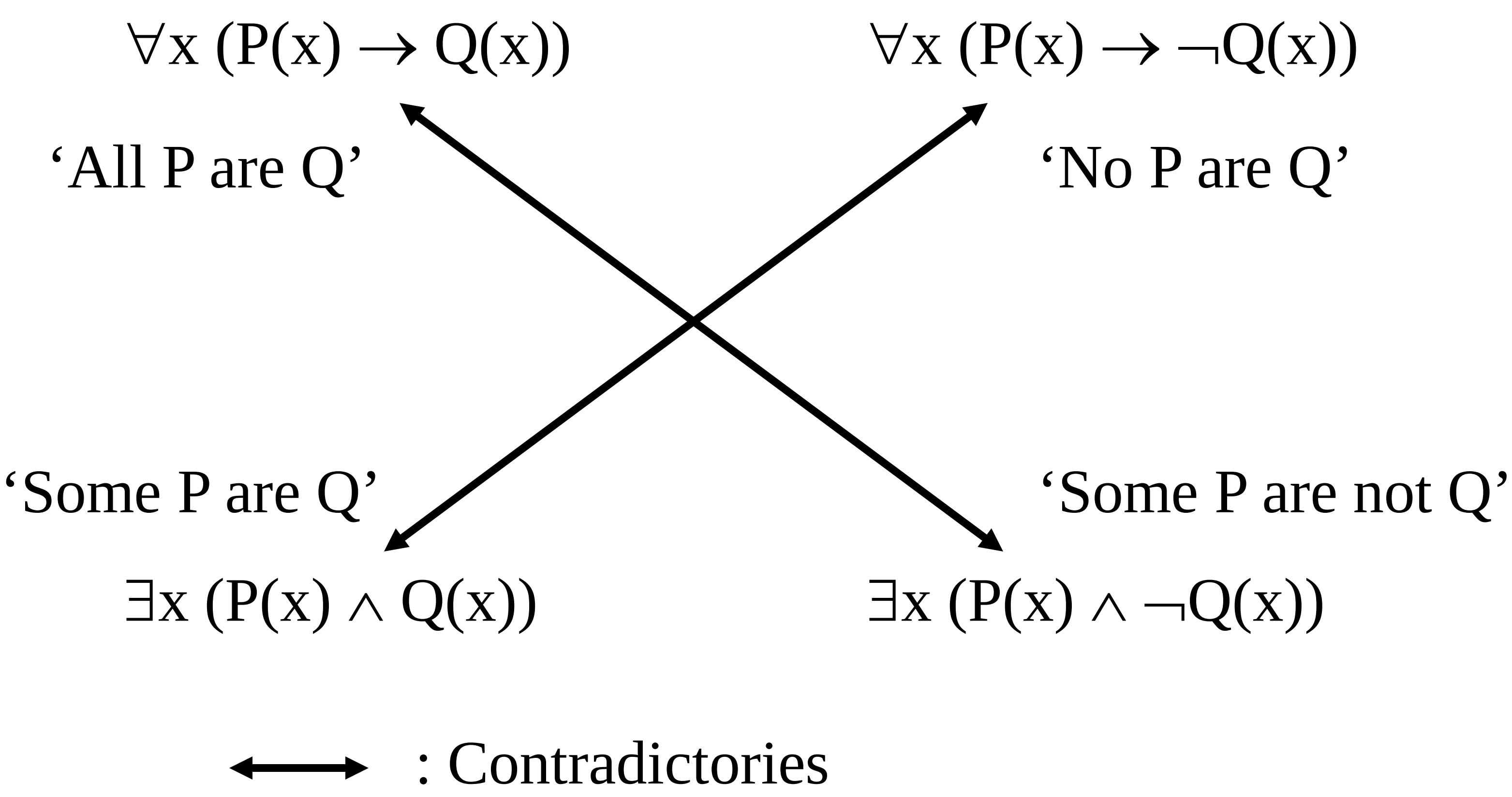
The Boolean Square Under the Assumption of Existential Import



Contraries: Can’t both be true

Subcontraries: Can’t both be false

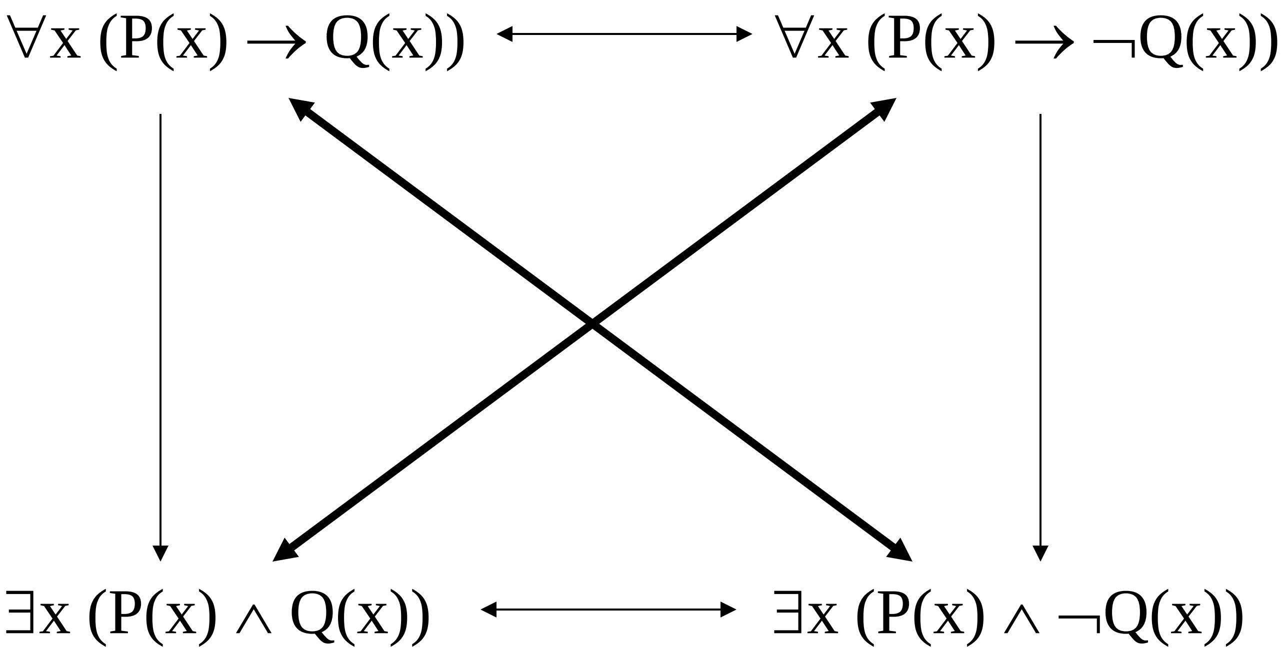
The Aristotelean Square of Opposition



The Assumption of Categorical Existential Import

- The Assumption of Categorical Existential Import is the assumption that for every property there is at least one thing that has that property.
- Under this assumption, $\exists x\ (P(x) \wedge Q(x))$ is true if $\forall x\ (P(x) \rightarrow Q(x))$ is true. Without the assumption, however, it's not: if nothing has property P, then $\exists x\ (P(x) \wedge Q(x))$ is false, even though $\forall x\ (P(x) \rightarrow Q(x))$ is (vacuously) true.
- In first-order logic, we do *not* make the assumption of categorical existential import. Thus, $\exists x\ (P(x) \wedge Q(x))$ is not considered a FO consequence of $\forall x\ (P(x) \rightarrow Q(x))$.

The Aristotelean Square Under the Categorical Assumption



Other Quantifier Equivalences

- $\forall$ over $\wedge$, and $\exists$ over $\vee$:
  - $\forall x\, (\varphi(x) \wedge \psi(x)) \Leftrightarrow \forall x\, \varphi(x) \wedge \forall x\, \psi(x)$
  - $\exists x\, (\varphi(x) \vee \psi(x)) \Leftrightarrow \exists x\, \varphi(x) \vee \exists x\, \psi(x)$
- Null Quantification:
  - $\forall x\, P \Leftrightarrow P$
  - $\exists x\, P \Leftrightarrow P$
- Replacing bound variables:
  - $\forall x\, \varphi(x) \Leftrightarrow \forall y\, \varphi(y)$
  - $\exists x\, \varphi(x) \Leftrightarrow \exists y\, \varphi(y)$
- Swapping quantifiers of same type:
  - $\forall x\, \forall y\, \varphi(x,y) \Leftrightarrow \forall y\, \forall x\, \varphi(x,y)$
  - $\exists x\, \exists y\, \varphi(x,y) \Leftrightarrow \exists y\, \exists x\, \varphi(x,y)$

Rewriting Example

If $\neg\forall x\,(P(x) \rightarrow Q(x))$ ('not all P's are Q's), then $\exists x\,(P(x) \wedge \neg Q(x))$ (some P's are not Q's), and vice versa:

$\neg\forall x\,(P(x) \rightarrow Q(x)) \Leftrightarrow$ (QN)
$\exists x\,\neg(P(x) \rightarrow Q(x)) \Leftrightarrow$ (Impl)
$\exists x\,(P(x) \wedge \neg Q(x))$