Propositional Logic Review

Computability and Logic

Boolean Connectives

Truth-Functional Connectives and Boolean Connectives

- Connectives are usually called *truth-functional* connectives:
  - This is because the *truth value* of a complex claim that has been constructed using a truth-functional connective is considered to be a *function* of the *truth values* of the claims that are being connected by that connective.
  - This is also why propositional logic is also called truth-functional logic.
- For now, we will focus on three connectives: and, or, not; these are called the Boolean connectives.

Truth-Table for Negation

P	$\neg P$
T	F
F	T

Truth-Table for Conjunction

P	Q	$P \wedge Q$
T	T	T
T	F	F
F	T	F
F	F	F

Truth-Table for Disjunction

P	Q	$P \vee Q$
T	T	T
T	F	T
F	T	T
F	F	F

Combining Complex Claims: Parentheses

- Using the truth-functional connectives, we can combine complex claims to make even more complex claims.
- We are going to use *parentheses* to indicate the exact order in which claims are being combined.
- Example: $(P \vee Q) \wedge (R \vee S)$ is a conjunction of two disjunctions.

Parentheses and Ambiguity

- An *ambiguous* statements is a statement whose meaning is not clear due to its syntax. Example : ”P or Q and R”
- In formal systems, an expression like $P \vee Q \wedge R$ is simply not allowed and considered *unsyntactical*.
- Claims in our formal language are therefore never ambiguous.
- One important application of the use of formal languages is exactly this: to avoid ambiguities!

Exclusive Disjunction vs Inclusive Disjunction

- Notice that the disjunction as defined by '$\vee$' is considered to be true if both disjuncts are true. This is called an *inclusive disjunction*.
- However, when I say "a natural number is either even or odd", I mean to make a claim that would be considered false if a number turned out to be both even and odd. Thus, I am trying to express an *exclusive disjunction*.

How to express Exclusive Disjunctions

- We could define a separate symbol for exclusive disjunctions, but we are not going to do that.
- Fortunately, exclusive disjunctions can be expressed using the symbols we already have: $(P \vee Q) \wedge \neg(P \wedge Q)$

P	Q	$(P \vee Q)$	$\wedge$	$\neg$	$(P \wedge Q)$
T	T	T	F	F	T
T	F	T	T	T	F
F	T	T	T	T	F
F	F	F	F	T	F

!

Conditionals

The Material Conditional

- Let us define the binary truth-functional connective ‘$\rightarrow$’ according to the truth-table below.
- The expression $P \rightarrow Q$ is called a *conditional*. In here, P is the *antecedent*, and Q the *consequent*.

P	Q	$P \rightarrow Q$
T	T	T
T	F	F
F	T	T
F	F	T

‘If … then …’ Statements

- The conditional is used to capture ‘if … then …’ statements.
- However, the match isn’t perfect.
- For example, we don’t want to say that the claim “If grass is green then elephants are big” is true just because grass is green and elephants are big, nor that any ‘if … then’ statement is automatically true once the ‘if’ part is false or the ‘then part true.
- The problem is that most English ‘if…then’ expressions aren’t meant to make a claim that is truth-functional in nature.
- Still, any ‘if … then …’ statement will be false if the ‘if’ part is true, but the ‘then’ part false, and the conditional captures at least this important truth-functional aspect of *any* ‘if … then …’ statement.
- So, while we will from now on refer to the conditional as an ‘if … then’ statement, we must be careful about the use of this, just as care must be taken when applying Newtonian physics to some situation.

Case in point: The Infamous ‘King-Ace’ problem

- The psychologist of reasoning gave the following logic problem to Princeton undergraduates:
  - Consider the following statement: “If there is a king in the hand, then there is an ace in the hand, or else if there is not a king in the hand, then there is an ace in the hand”. What follows from this statement?
- Almost all students responded that it can be inferred that there is an ace in the hand.
- Johnson-Laird, however, said that what can be concluded is that there is *not* an ace in the hand, and that this is evidence that people can easily make logical reasoning mistakes! …. Really?

Necessary and Sufficient Conditions

- *Sufficient Condition*: Something (P) is a sufficient condition for something else (Q) iff P being the case guarantees (i.e. is sufficient) Q being the case. In logic: $P \rightarrow Q$
- *Necessary Condition*: Something (P) is a necessary condition for something else (Q) iff P being the case is required (i.e. is necessary) for Q being the case. In logic: $Q \rightarrow P$

‘If’ vs ‘Only if’

- Sufficient conditions are expressed in English using ‘if’, while necessary conditions are expressed using ‘only if’.
- Thus:
  - ‘If P then Q’: $P \rightarrow Q$
  - ‘P if Q’: $Q \rightarrow P$
  - ‘P only if Q’: $P \rightarrow Q$
  - ‘Only if P, Q’: $Q \rightarrow P$

‘Unless’

- A statement of the form ‘P unless Q’ usually means: ‘P is the case as long as Q is not the case, but if Q is the case, then P is not the case’.
- However, the last part is not always intended. That is, sometimes we say ‘P unless Q’ to mean ‘P is the case as long as Q is not the case. However, if Q is the case, then I don’t know about P’.
  - Example: If I say: “You are not going to pass the final unless you study hard”, I mean that if you don’t study, you are not going to pass the final, but I don’t mean that if you do study, you will pass the final!
- For this reason, we are going to translate ‘P unless Q’ with just $\neg Q \rightarrow P$ unless stated otherwise.

‘If and only if’ and the Material Biconditional

- A statement of the form ‘P if and only if Q’ (or ‘P iff Q’) is short for ‘P if Q, and P only if Q’. Hence, we could translate this as $(P \rightarrow Q) \wedge (Q \rightarrow P)$. However, since this is a common expression, we define a new connective ‘$\leftrightarrow$’:

P	Q	$P \leftrightarrow Q$
T	T	T
T	F	F
F	T	F
F	F	T

Necessary and Sufficient Conditions Revisited: Definitions

- A bachelor is an unmarried (adult) male.
- So, being unmarried is a necessary condition for being a bachelor. So: $B \rightarrow U$ (but *not*: $U \rightarrow B$)
- And, being male is a necessary condition for being a bachelor: $B \rightarrow M$ (*not* $M \rightarrow B$)
- Being unmarried *and* male is sufficient to be a bachelor: $(U \wedge M) \rightarrow B$
- So, they are (each) necessary and (together) sufficient: $B \leftrightarrow (U \wedge M)$, i.e. you are a bachelor if and only if you are unmarried and male.
- To define something, we often try and provide the necessary and sufficient conditions for that something.

Logical Properties

Truth Tables

- Truth-tables can be used for:
  - *defining* the truth-conditions of truth-functional connectives
  - *evaluating* the truth-conditions of any complex statement

Tautologies

- A tautology is a statement that is necessarily true.
- Example: $P \vee \neg P$

P	$P \vee \neg P$
T	T T F T
F	F T T F

Contradictions

- A contradiction is a statement that is necessarily false.
- Example: $P \wedge \neg P$

P	$P \wedge \neg P$
T	T F F T
F	F F T F

Contingencies

- A contingency is a statement that can be true as well as false
- Example: P

P	P
T	T
F	F

Equivalences

- Two statements are equivalent if they have the exact same truth-conditions.
- Example: P and ¬¬P

P	P	¬¬P
T	T	T F T
F	F	F T F

Contradictories

- Two statements are contradictories if one of them is false whenever the other one is true and vice versa.
- Example: P and $\neg$P

P	P	$\neg$P
T	T	F T
F	F	T F

Implication

- One statement implies a second statement if it is impossible for the second statement to be false whenever the first statement is true.
- Example: P implies $P \vee Q$

P	Q	P	$P \vee Q$
T	T	T	T
T	F	T	T
F	T	F	T
F	F	F	F

Consistency

- A set of statements is consistent if it is possible for all of them to be true at the same time.
- Example: $\{P, P \vee Q, \neg Q\}$

P	Q	P	$P \vee Q$	$\neg Q$
T	T	T	T	F
T	F	T	T	T
F	T	F	T	F
F	F	F	F	T

Consequence

- A statement is a consequence of a set of statements if it is impossible for the statement to be false while each statement in the set of statements is true.
- Example: P is a consequence of $\{P \vee Q, \neg Q\}$

P	Q	$P \vee Q$	$\neg Q$	P
T	T	T	F	T
T	F	T	T	T
F	T	T	F	F
F	F	F	T	F

Validity

- An argument is valid if it is impossible for the conclusion to be false whenever all of its premises are true.
- Example: $P \vee Q, \neg Q \therefore P$

P	Q	$P \vee Q$	$\neg Q$	P
T	T	T	F	T
T	F	T	T	T
F	T	T	F	F
F	F	F	T	F

Implication, Consequence, Validity

- The notions of implication, consequence, and validity are very closely related:
- A statement φ implies a statement ψ if and only if ψ is a consequence of the set of statements $\{\varphi\}$
- For implication and consequence we use the symbol '$\Rightarrow$':
  - If statement φ implies statement ψ we write $\varphi \Rightarrow \psi$
  - If statement ψ is a consequence of a set of statements $\{\varphi_1, \ldots, \varphi_n\}$, we write $\{\varphi_1, \ldots, \varphi_n\} \Rightarrow \psi$
- An argument consisting of premises $\varphi_1, \ldots, \varphi_n$ and conclusion ψ is valid iff $\{\varphi_1, \ldots, \varphi_n\} \Rightarrow \psi$
- The terms implication, consequence and validity can therefore be used interchangeably.