Formal Proofs for Quantifiers in F

Computability and Logic

Quantifier Rules in *F*

- There are 4 quantifier rules in *F*:
  - Universal Introduction and Elimination
  - Existential Introduction and Elimination
- As we saw last time, Universal Introduction and Existential Elimination have restrictions in that the rules cannot be applied relative to just any individual constant. The system *F* deals with those restrictions through the use of subproofs. We'll see later how that works.
- Fortunately, Universal Elimination and Existential Introduction do not have any restrictions, so we'll start with those.

Notation

- In describing the rules, the following notation is useful:
  - $\varphi(x)$ is a wff with zero or more instances of x as the only free variable.
  - $\varphi(a/x)$ is the statement that results when substituting ‘a’ for all occurrences of ‘x’ that are free in $\varphi(x)$.
  - If it is clear which variable we are substituting, we will simply write $\varphi(a)$.

∀ Elim

- Universal Elimination (∀ Elim) allows one to conclude that any thing has a certain property if everything has that property:

$$\forall x\ \varphi(x)$$

$$\vdots$$

$$\varphi(a)$$

Good and Bad Uses of ∀ Elim

Good

$\forall x\ SameSize(x,x)$

⋮

$SameSize(a,a)$

Bad

$\forall x\ SameSize(x,x)$

⋮

$SameSize(a,b)$

→ The *same* individual constant should be used!

Bad

$\forall x\ SameSize(x,x)$

⋮

$SameSize(x,a)$

→ *All* free occurrences of x should be replaced!

Bad

$\forall x\ (Tet(x) \rightarrow \forall x\ Large(x))$

⋮

$Tet(a) \rightarrow \forall x\ Large(a)$

→ Only *free* occurrences of x should be replaced!

∃ Intro

- Existential Introduction (∃ Intro) allows one to conclude that something has a certain property if some thing has that property:

$$\begin{array}{|l} \varphi(a) \\ \vdots \\ \exists x\, \varphi(x) \end{array}$$

Good and Bad Uses of ∃ Intro

Good

SameSize(a,a)
⋮
∃x SameSize(x,x)

Good

SameSize(a,a)
⋮
∃x SameSize(a,x)

→ Not all occurrences of a have to be replaced!

Bad

SameSize(a,b)
⋮
∃x SameSize(x,x)

→ The *same* individual constant should be used!

Bad

∃x SameSize(a,x)
⋮
∃x ∃x SameSize(x,x)

→ Doesn't follow the rule (no free x's in ∃x SameSize(x,x))

∀ Intro

- Universal Introduction (∀ Elim) allows one to conclude that everything has a certain property if anything has that property:

$$\begin{array}{l|l} & \begin{array}{|l} \triangleright a \\ \quad \vdots \\ \varphi(a) \end{array} \\ & \forall x\ \varphi(x) \end{array}$$

a may not occur before the subproof, unless all subproofs in which it occurs have been closed. a may not occur in φ(x) either.

Good and Bad Uses of ∀ Intro

Good

$\triangleright$a
⋮
SameSize(a,a)
∀x SameSize(x,x)

Still Good

$\triangleright$a
⋮
Tet(a)
∀x Tet(x)

→ a occurs outside subproof, but only in a subproof that has been closed.

Bad

Tet(a)
$\triangleright$a
⋮
SameSize(a,a)
∀x SameSize(x,x)

→ a occurs before subproof!

Bad

$\triangleright$a
⋮
SameSize(a,a)
∀x SameSize(a,x)

→ a occurs in SameSize(a,x)!

∃ Elim

- Existential Elimination (∃ Elim) allows one to conclude anything that follows from some thing having a certain property, given that something has that property.

| $\exists x\ \varphi(x)$
| | $\triangleright a\ \varphi(a)$
| | ⋮
| | Q
| Q

a may not occur before the subproof, unless all subproofs in which it occurs have been closed. a may not occur in Q either.

Good and Bad Uses of ∃ Elim

Good

$\exists x$ SameSize(x,x)

$\triangleright$a SameSize(a,a)

⋮

$\exists x$ Cube(x)

$\exists x$ Cube(x)

Still Good

$\exists x$ Cube(x)

$\triangleright$a Cube(a)

⋮

Tet(b)

Tet(b)

→ a occurs before subproof, but only in a subproof which has been closed.

Bad

Tet(a)

⋮

$\exists x$ SameSize(x,x)

$\triangleright$a SameSize(a,a)

⋮

$\forall x$ Large(x)

$\forall x$ Large(x)

→ a occurs before subproof!

Bad

$\triangleright$a SameSize(a,a)

⋮

Large(a)

Large(a)

→ a occurs in Large(a)!

General Conditional Proof

- Most universal claims are proven by the application of ∀ Intro. Also, most universal claims are of the form $\forall x\,(\varphi(x) \rightarrow \psi(x))$. Thus, most proofs of universal claims would look like this:

$$
\begin{array}{l}
\quad \triangleright a \\
\quad\quad \varphi(a) \\
\quad\quad \vdots \\
\quad\quad \psi(a) \\
\quad \varphi(a) \rightarrow \psi(a) \\
\forall x\,(\varphi(x) \rightarrow \psi(x))
\end{array}
$$

General Conditional Proof (Continued)

- Because this is such a common pattern, the rule of General Conditional Proof allows us to take a little short cut:

$$
\begin{array}{l|l}
 & \triangleright a \;\varphi(a) \\
 & \quad \vdots \\
 & \psi(a) \\
\hline
\forall x\,(\psi(x) \rightarrow \varphi(x)) &
\end{array}
$$

a may not occur before the subproof, unless all subproofs in which it occurs have been closed. a may not occur in $\varphi(x) \rightarrow \psi(x)$ either.