Truth Trees for Predicate Logic

Computability and Logic

Running Examples

Valid Argument (13.24):
$\exists x\ (\mathrm{Cube}(x) \land \mathrm{Small}(x))$
$\therefore \exists x\ \mathrm{Cube}(x) \land \exists x\ \mathrm{Small}(x)$

Invalid Argument (13.25):
$\exists x\ \mathrm{Cube}(x) \land \exists x\ \mathrm{Small}(x)$
$\therefore \exists x\ (\mathrm{Cube}(x) \land \mathrm{Small}(x))$

Truth-Functional Expansions

- Suppose that our Universe of Discourse (UD) contains only the objects a and b.
- Given this UD, the claim $\forall x$ Cube(x) is true iff Cube(a) $\wedge$ Cube(b) is true.
- Similarly, the claim $\exists x$ Cube(x) is true iff Cube(a) $\vee$ Cube(b) is true.
- The truth-functional interpretation of the FO statements given a fixed UD is called the *truth-functional expansion* of the original FO statement with regard to that UD.

Truth-Functional Expansions and Proving FO Invalidity

- Truth-Functional expansions can be used to prove FO invalidity. Example (13.25):

$\exists x \text{ Cube}(x) \land \exists x \text{ Small}(x)$
$\therefore \exists x \text{ (Cube}(x) \land \text{Small}(x))$

UD = {a,b}

T T F T F T T
$(\text{Cube}(a) \lor \text{Cube}(b)) \land (\text{Small}(a) \lor \text{Small}(b))$
$\therefore (\text{Cube}(a) \land \text{Small}(a)) \lor (\text{Cube}(b) \land \text{Small}(b))$
T F F F F F T

This shows that there is a world in which the premise is true and the conclusion false. Hence, the original argument is FO invalid.

Truth-Functional Expansions and Proving FO Validity

- If the truth-functional expansion of an FO argument in some UD is truth-functionally invalid, then the original argument is FO invalid, but if it is truth-functionally valid, then that does not mean that the original argument is FO valid.
- For example, with UD = {a}, the expansion of the argument would be truth-functionally valid. In general, it is always possible that adding one more object to the UD makes the expansion invalid.
- Thus, *we can’t prove validity using the expansion method*, as we would have to show the expansion to be valid in every possible UD, and there are infinitely many UD’s.
- The expansion method is therefore only good for proving invalidity. Indeed, it searches for countermodels.

The Expansion Method as a Systematic Procedure

- Still, the expansion method can be made into a systematic procedure to test for FO invalidity:
  - Step 1: Expand FO argument (which can be done systematically) in UD = {a}.
  - Step 2: Use some systematic procedure (e.g. truth-table method or truth-tree method) to test whether the expansion is TF invalid. If it is TF invalid, then stop: the FO argument is FO invalid. Otherwise, expand FO argument in UD = {a,b}, and repeat step 2.

Incompleteness of the Expansion Method

- We saw that the expansion method is not a complete test for FO validity.
- However, it is also an incomplete test for FO invalidity!
- Proof: Consider the following argument:

$\forall x \forall y(x \neq y \rightarrow ((x>y \vee y>x) \wedge$
$\neg(x>y \wedge y>x)))$
$\forall x \forall y \forall z((x>y \wedge y>z) \rightarrow x>z)$
$\therefore \exists x \forall y(x \neq y \rightarrow x>y)$

For any UD with an arbitrarily large yet finite number of objects, the expansion of this argument will be truth-functionally valid. However, the argument is FO invalid (consider the natural numbers)!

A More Focused Search

- A further drawback of the expansion method is that the search for a counterexample is very inefficient.
- A focused search for a counterexample is more efficient:
  - (13.25) I want there to be at least one cube, and at least one small object, but no small cubes. So, if we have a cube, a, then a cannot be small, so I need a second object, b, which is small, but not a cube. Counterexample, so the argument is invalid.

Advantage of a Focused Search

- The focused search method is like the indirect truth-table method.
- Indeed, like the indirect truth-table method, the focused search method can prove validity:
  - (13.24) I want there to be at least one small cube. Let us call this small cube a. How, I don’t want it to be true that there is at least one cube and at least one small object. However, a is both a cube and small. Contradiction, so I can’t generate a counterexample.

Truth-Trees for Predicate Logic

- Like the direct method, the focused search method needs to be systematized, especially since the search often involves making choices.
- Fortunately, the truth-tree method, which systematized the indirect truth-table method in truth-functional logic, can be extended for predicate logic.

Truth-Tree Rules for Quantifiers

$\neg\forall x\ \varphi(x)$ √

$\exists x\ \neg\varphi(x)$

$\neg\exists x\ \varphi(x)$ √

$\forall x\ \neg\varphi(x)$

$\exists x\ \varphi(x)$ √

$\varphi(c)$

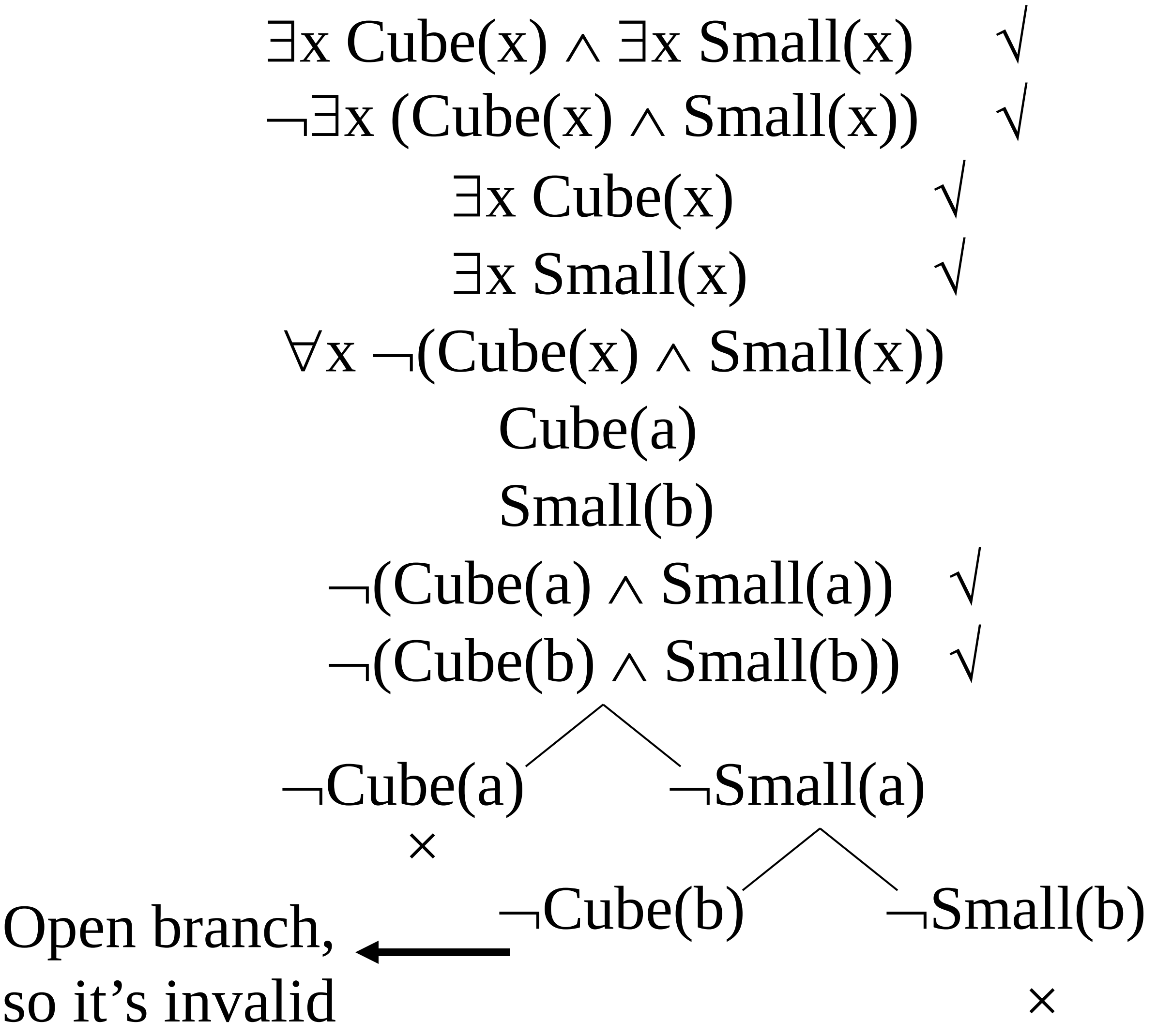
with ‘c’ a
new constant
in that branch

$\forall x\ \varphi(x)$

$\varphi(c)$

with ‘c’ any
constant

Truth-Tree Example I



$\exists x\, \mathrm{Cube}(x) \land \exists x\, \mathrm{Small}(x)$ √

$\neg\exists x\, (\mathrm{Cube}(x) \land \mathrm{Small}(x))$ √

$\exists x\, \mathrm{Cube}(x)$ √

$\exists x\, \mathrm{Small}(x)$ √

$\forall x\, \neg(\mathrm{Cube}(x) \land \mathrm{Small}(x))$

$\mathrm{Cube}(a)$

$\mathrm{Small}(b)$

$\neg(\mathrm{Cube}(a) \land \mathrm{Small}(a))$ √

$\neg(\mathrm{Cube}(b) \land \mathrm{Small}(b))$ √

$\neg\mathrm{Cube}(a)$ $\neg\mathrm{Small}(a)$

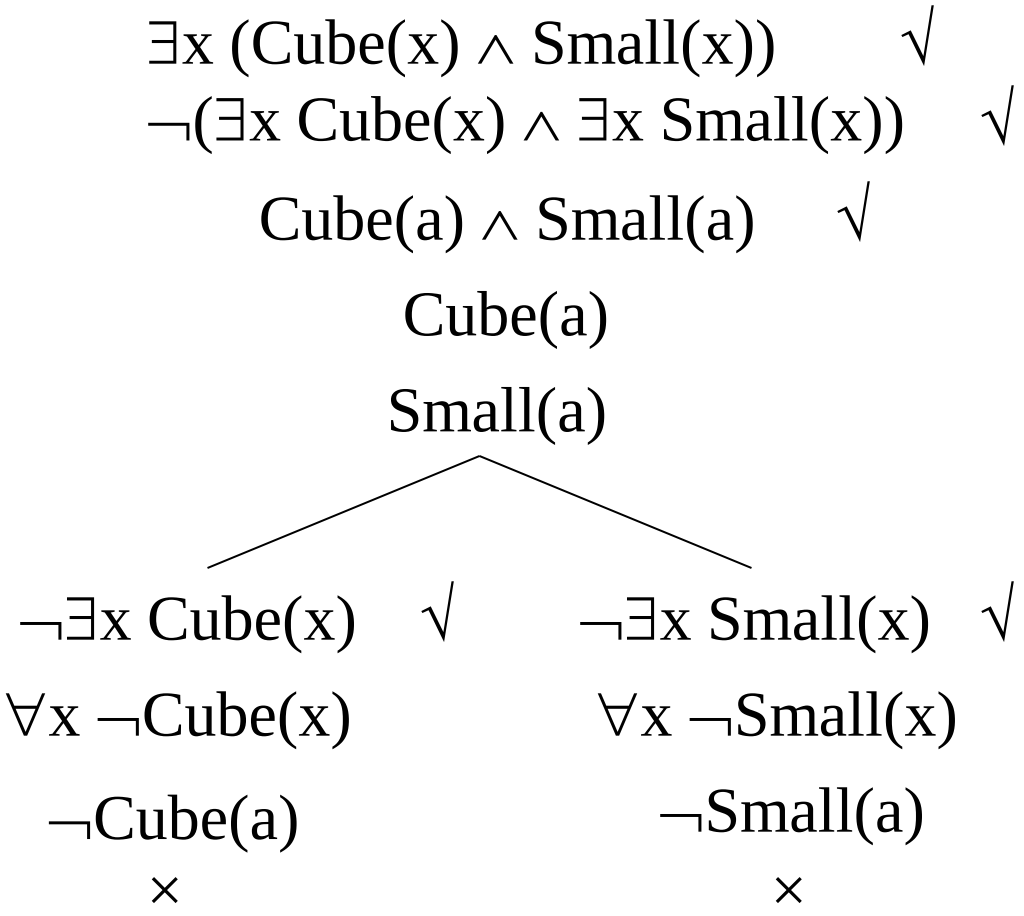
×

$\neg\mathrm{Cube}(b)$ $\neg\mathrm{Small}(b)$

×

Open branch,
so it’s invalid

Truth-Tree Example II



$\exists x\,(\text{Cube}(x) \land \text{Small}(x))$ √

$\neg(\exists x\,\text{Cube}(x) \land \exists x\,\text{Small}(x))$ √

$\text{Cube}(a) \land \text{Small}(a)$ √

$\text{Cube}(a)$

$\text{Small}(a)$

$\neg\exists x\,\text{Cube}(x)$ √

$\forall x\,\neg\text{Cube}(x)$

$\neg\text{Cube}(a)$

×

$\neg\exists x\,\text{Small}(x)$ √

$\forall x\,\neg\text{Small}(x)$

$\neg\text{Small}(a)$

×

All branches close,
so it’s valid

Finished Trees

- A branch is closed if it contains a statement and its negation.
- An open branch is finished if every statements in that branch that has not been decomposed is either a literal or a universal that has been instantiated for every constant in that branch.
- A tree is finished if all its branches are closed (in which case the statements at the root cannot be satisfied), or if it contains a finished open branch (in which case the statements can be satisfied).

Infinite Trees

$\forall x\ \exists y\ \text{Likes}(x,y)$

$\exists y\ \text{Likes}(a,y)$ √

$\text{Likes}(a,b)$

$\exists y\ \text{Likes}(b,y)$ √

$\text{Likes}(b,c)$

$\exists y\ \text{Likes}(c,y)$ √

$\text{Likes}(c,d)$

$\exists y\ \text{Likes}(d,y)$ √

$\text{Likes}(d,e)$

⋮

This tree will never be finished, so the tree method will not give us any answer!